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# NUMERICAL SOLUTION OF SATELLITE MOTION UNDER THE EFFECTS OF THE EARTH'S GRAVITY OF THE AXIAL SYMMETRY AND DRAG FORCE IN TERMS OF THE KS-REGULARIZED VARIABLES

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**Abstract**

This paper is concerned with an orbit prediction using one of the best regular theories (KS-regularized variables). Perturbations due to the Earth's gravitational field with axial symmetry up to the fourth order zonal harmonic and rotating/non-rotating atmosphere (variation in density model with height) are considered. Applications of the problem will be illustrated by numerical and graphical example.

## 1. Introduction

It is well known that the solutions of the Classical Newtonian Equations of motion are unstable and these equations are not suitable for long-term integrations. Many transformations have emerged in the literature in the recent past to stabilize the equations of motion either to reduce the accumulation of local numerical errors or allowing of using a larger integration step size, in the transformed space, or both.

Examples of such transformations include the use of a new independent variable-time transformation, transformation to orbital parameter space which tends to decouple fast and slow variables, and the use of integrals as control terms. One of such transformation, known as the KS-transformation, is due to Kustaa-neimo and Stiefel, who regularized the non-linear Kepler motion and reduced it to linear differential equations of a harmonic oscillator of constant frequency. Stiefel and Scheifele, (1971) further developed the application of the KS-transformation to problems of perturbed motion, producing a perturbational equations version.

## 2. Formulate the Problem:

The equations of motion of an artificial satellite are given generally as

$$\ddot{\vec{x}} + \frac{\mu}{r^3}\vec{x} = -\frac{\partial V}{\partial \vec{x}} + \vec{P}, \tag{2.1}$$

where $\vec{x}$ is the position vector in a rectangular frame (the physical frame), $r = |\vec{x}|$ is the distance from the origin, $\mu$ is the Earth's gravitational constant, $V$ is the perturbed time independent potential and $\vec{P}$ is the resultant of all non-conservative perturbing forces and forces derivable from a time dependent potential.

The potential of the Earth's gravity with axial symmetry can be written as

$$V = \mu \sum_{i=2}^{\infty} R^i \; J_i \left(\frac{1}{r}\right)^{i+1} P_i(x_3/r)\,, \tag{2.2}$$

where $R$ is the Earth's equatorial radius, $J_i$ is the non-dimensional coefficient of the Earth's oblateness and $P_i(x_3/r)$ is the Legendre polynomial of order $i$. In the present paper we shall assume that the potential of the Earth's gravity of the axial symmetry is taken up to the fourth order zonal harmonics $J_4$, then Eq.(2.2) rewrite as

$$\begin{aligned} V = {} & \frac{3}{2} Q_2\, x_3^2\, r^{-5} - \frac{1}{2} Q_2\, r^{-5} + \frac{5}{2} Q_3\, x_3^3\, r^{-7} - \frac{3}{2} Q_3\, x_3\, r^{-5} + \\ & \frac{35}{8} Q_4\, x_3^4\, r^{-9} - \frac{15}{4} Q_4\, x_3^2\, r^{-7} + \frac{3}{8} Q_4\, r^{-5}\,, \end{aligned} \tag{2.3}$$

where $\quad Q_i = \mu\, R^i\, J_i\,,\quad$ i = 2(1)4

and $\quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}\,.$

Since the perturbing acceleration due to air drag is expressed as

$$\vec{D} = -\frac{1}{2} C_D\, \frac{A}{M}\, \rho\, \left|\vec{v}\right|\, \vec{v} \tag{2.4}$$

where - $C_D$ is the non-dimensional drag coefficient depending on the satellite geometry and in most cases its value lies between 2.1 & 2.3;

- $A$ is the effective cross-sectional area, $M$ is the satellite mass;
- $\rho$ is the density function of the ambient gas (the atmosphere) and depends primarily on the altitude and to a lesser extent on the solar and geomagnetic activity. In this paper we'll test two models of air density which are (Bakry and Hassan, 2005)

$$\text{1- } \rho = \rho_0\, e^{[-(r-r_p)/h]}\,, \qquad \text{(model I)} \tag{2.5}$$

where $\rho_0$ is the value of $\rho$ at the reference level $r_p$, and $p$ is the suffix refers to perigee.

$$\text{2- } \rho = \rho_0 \left[\frac{r_0 - R_\oplus}{r - R_\oplus}\right]^{\tau}, \qquad \text{(model II)} \tag{2.6}$$

where $\rho_0$ is the value of $\rho$ at the reference level $r_0$, while $R_\oplus$ and $\tau$ are two adjustable parameters. They can be adapted to the estimated or observed variations of the solar activity and periodically updated so that the dynamics of the atmosphere is taken into account. The value of $R_\oplus$ is approximately equal to the mean Earth's equatorial radius and $\tau$ equals the inverse of gradient of the density scale height and can take values in the range from 3 to 9 (Delhaise, 1991).

- $\vec{v}$ is the velocity of the satellite relative to the atmosphere and is generally computed through

$$\vec{v} = \vec{v}^{\,*} - \vec{\omega} \times \vec{r} \,, \tag{2.7}$$

where $\vec{v}^{\,*}$ is the velocity of the satellite with respect to the Earth's center, i.e., ($v_1 = \dot{x}_1$, $v_2 = \dot{x}_2$, $v_3 = \dot{x}_3$); and $\vec{\omega}$ is the west-to-east angular velocity vector of the atmosphere (Bakry and Hassan, 2002; Bakry and Hassan, 2005).

Finally, the equations of motion of an artificial satellite in KS-regularized variables are

$$\vec{u}'' + \alpha_k \vec{u} = \frac{r}{2} \vec{\lambda} \,, \tag{2.8.1}$$

$$\alpha_k = -<\vec{u}',\vec{\lambda}> , \tag{2.8.2}$$

$$t' = r \,, \tag{2.8.3}$$

$$r'' + 4\alpha_k r = \mu + r <\vec{u},\vec{\lambda}> , \tag{2.8.4}$$

hence $< \vec{a},\vec{b} >$ is used to denote the scalar product of two vectors $\vec{a}$ and $\vec{b}$ ;

where - $\alpha_k$ is one-half of the negative Keplerian energy as

$$\alpha_k = \left( \frac{\mu}{2} - <\vec{u}',\vec{u}'> \right) / r \,;$$

- $$\vec{\lambda} = L^T(\vec{u})\vec{b} = L^T(\vec{u}) \left( -\frac{\partial V}{\partial \vec{x}} + \vec{P} \right),$$

- $$L(\vec{u}) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix},$$

and - $r = -<\vec{u},\vec{u}>$, $\quad r' = 2<\vec{u},\vec{u}'>$;

Denoting differentiation with respect to the new time *s* (knowing as the fictitious time) by a prime ($'$), since the independent variable is changed from time ($t$) to fictitious time ($s$) according to (Stiefel and Scheifele, 1971)

$$dt/ds = r\,,$$

then for any variable $\zeta$ we have

$$\zeta' = r\,\dot{\zeta}\,.$$

### 3. Equations of Motion:

The differential equations of motion for the satellite in KS-regularized variables under the perturbations of the Earth's gravity and air drag are

$$u_1'' = -\alpha_k\, u_1 + \frac{r}{2}\,\lambda_1\,, \tag{3.1}$$

$$u_2'' = -\alpha_k\, u_2 + \frac{r}{2}\,\lambda_2\,, \tag{3.2}$$

$$u_3'' = -\alpha_k\, u_3 + \frac{r}{2}\,\lambda_3\,, \tag{3.3}$$

$$u_4'' = -\alpha_k\, u_4 + \frac{r}{2}\,\lambda_4\,, \tag{3.4}$$

$$\alpha_k' = -u_1'\,\lambda_1 - u_2'\,\lambda_2 - u_3'\,\lambda_3 - u_4'\,\lambda_4\,, \tag{3.5}$$

$$t' = r\,, \tag{3.6}$$

$$r'' = \mu + r\left(-4\,\alpha_k + u_1\,\lambda_1 + u_2\,\lambda_2 + u_3\,\lambda_3 + u_4\,\lambda_4\right), \tag{3.7}$$

where $\lambda_1 = \ \ u_1\, b_1 + u_2\, b_2 + u_3\, b_3\,,$

$$\lambda_2 = -\,u_2\, b_1 + u_1\, b_2 + u_4\, b_3\,,$$

$$\lambda_3 = -\,u_3\, b_1 - u_4\, b_2 + u_1\, b_3\,,$$

$$\lambda_4 = \ \ u_4\, b_1 - u_3\, b_2 + u_2\, b_3\,;$$

$$b_1 = \frac{15}{2} Q_2\, x_1\, x_3^2\, r^{-4} - \frac{3}{2} Q_2\, x_1\, r^{-5} + \frac{35}{2} Q_3\, x_1\, x_3^3\, r^{-9} - \frac{15}{2} Q_3\, x_1\, x_3\, r^{-7} + \frac{315}{8} Q_4\, x_1\, x_3^4\, r^{-11} - \frac{105}{4} Q_4\, x_1\, x_3^2\, r^{-9} + \frac{15}{8} Q_4\, x_1\, r^{-7} - \gamma\, \rho\, v\, v_1\,,$$

$$b_2 = \frac{15}{2} Q_2\, x_2\, x_3^2\, r^{-4} - \frac{3}{2} Q_2\, x_2\, r^{-5} + \frac{35}{2} Q_3\, x_2\, x_3^3\, r^{-9} - \frac{15}{2} Q_3\, x_2\, x_3\, r^{-7} +$$
$$\frac{315}{8} Q_4\, x_2\, x_3^4\, r^{-11} - \frac{105}{4} Q_4\, x_2\, x_3^2\, r^{-9} + \frac{15}{8} Q_4\, x_2\, r^{-7} - \gamma\, \rho\, v\, v_2\,,$$

$$b_3 = -\frac{9}{2} Q_2\, x_3\, r^{-5} + \frac{15}{2} Q_2\, x_3^3\, r^{-7} - 15\; Q_3\, x_3^2\, r^{-7} + \frac{35}{2} Q_3\, x_3^4\, r^{-9} + \frac{3}{2} Q_3\, r^{-5}$$
$$- \frac{175}{4} Q_4\, x_3^3\, r^{-9} + \frac{315}{8} Q_4\, x_3^5\, r^{-11} + \frac{75}{8} Q_4\, x_3\, r^{-7} - \gamma\, \rho\, v\, v_3\,;$$

and $\qquad \gamma = -\frac{1}{2} C_D \frac{A}{M}$ ,

since $\rho$ can take any of the above two models of air density with non-rotating atmosphere, then the velocity $\vec{v}$ becomes equal $\vec{v}^{\,*}$, but with rotating atmosphere then the velocity $\vec{v}$ becomes

$$v_1 = \dot{x}_1 + \omega_3\, x_2\,, \quad v_2 = \dot{x}_2 - \omega_3\, x_1\,, \quad v_3 = \dot{x}_3\,,$$

with neglecting the two components $(\omega_1, \omega_2)$ of the angular velocity of the atmosphere. Because of their values are very very small.

**4. Solution Technique:**

In this section, the solution technique of the formulations of section 3 will be applied by tow steps. The first step is to transform Eqs.(3.1) up to Eqs.(3.7) into first order differential equations by the following substitutions

$y_i = u_i$, $y_{i+4} = u_i'$, i = 1(1)4, $y_9 = \alpha_k$, $y_{10} = t$, $y_{11} = r$ and $y_{12} = r'$.

Then the first order system of the problem becomes

$$y_1' = y_5\,, \qquad (4.1) \qquad\qquad y_2' = y_6\,, \qquad (4.2)$$

$$y_3' = y_7\,, \qquad (4.3) \qquad\qquad y_4' = y_8\,, \qquad (4.4)$$

$$y_5' = -y_9\, y_1 + \tfrac{1}{2} y_{11}\, b_1\,, \qquad (4.5)$$

$$y_6' = -y_9\, y_2 + \tfrac{1}{2} y_{11}\, b_2\,, \qquad (4.6)$$

$$y_7' = -y_9\, y_3 + \tfrac{1}{2} y_{11}\, b_3\,, \qquad (4.7)$$

$$y_8' = -y_9\, y_4 + \tfrac{1}{2} y_{11}\, b_4\,, \qquad (4.8)$$

$$y_9' = -y_5\, b_1 - y_6\, b_2 - y_7\, b_3 - y_8\, b_4\,, \qquad (4.9)$$

$$y'_{10} = y_{11}, \tag{4.10}$$

$$y'_{11} = y_{12}, \tag{4.11}$$

$$y'_{12} = \mu + y_{11}\left(y_1 b_1 + y_2 b_2 + y_3 b_3 + y_4 b_4 - 4 y_9\right). \tag{4.12}$$

The second step is solving the above system by using the fourth-order Runge-Kutta method with a fixed step size. So, the initial values, the step size and the accuracy checks, which we need in the solution, were derived in Bakry and Hassan, (2002) and Bakry and Hassan, (2005).

**5. Results and Conclusion:**

We'll take as the numerical example the Indian Satellite (RS-1) at 300 Km height which was launched from Sriharikota range on 18 July, 1980 and remained in its orbit for around 371 days or about 4000 revolutions (Sharma and Mani, 1985). The initial position and velocity components are

$$\vec{x}_0 = (1626.742,\ 6268.094,\ -1776.018)\ Km,$$

$$\dot{\vec{x}}_0 = (-5.920522,\ 0.239214,\ -5.15883)\ Km/sec,$$

where one orbital revolution is elapsed $1^h.588352058$ and its area equals $0.319019\ m^2$ and its mass equals $35.443\ Kg$.

Since the adopted physical constant are

$$R = 6378.135\ Km, \qquad \mu = 398600.8\ Km^3/sec^2,$$

and the coefficients of the four order zonal harmonic are

$$J_2 = 1.0826157 \times 10^{-3}, \quad J_3 = -2.53648 \times 10^{-6}, \quad J_4 = -1.6233000 \times 10^{-6},$$

where $C_D = 2.2$ and $\omega_3 = 7.292115833 \times 10^{-5}$ *rad./sec.* (Sharaf and Awad, 1985), and we'll chose $\tau$ (the arbitrary const.) equals 4.

We'll use all the above values to compute the position and velocity components, i.e., the six elements; especially the semi-major axis, the eccentricity and inclination, because of these elements are strongly affected by our concerned forces. Also, we'll get the accuracy check tables (bilinear relation, *BI*) at any time. These are illustrated in the following figures and tables over one thousand revolutions.

All the figures show the effects of the Earth's gravitational field with axial symmetry up to the four order zonal harmonic and air drag with and/or without rotating atmosphere. These effects are big and clear in the model II of air density (Figs. 2) than model I (Figs.1), because of the value of the term $\left[(r_0 - R_\oplus)/(r - R_\oplus)\right]^4$ is greater than the value of $e^{[-(r-r_p)/h]}$, so we can deduced that the model II is more accurate than the model I, because of their prediction data are approximately coincide with the observed one (Sharma and Mani, 1985).

Also, all the Figures show a small difference between the cases with and/or without consideration of the rotation of the atmosphere. This difference is due to the small value of $\omega_3$.

All Tables give the bilinear relation (*BI*) for all the studied cases, which indicates a good prediction for the numerical solution.

The numerical results are just only as an example, since this method could be applied to any orbit.

**Fig.(1-a): one thousand revolutions (model I).**

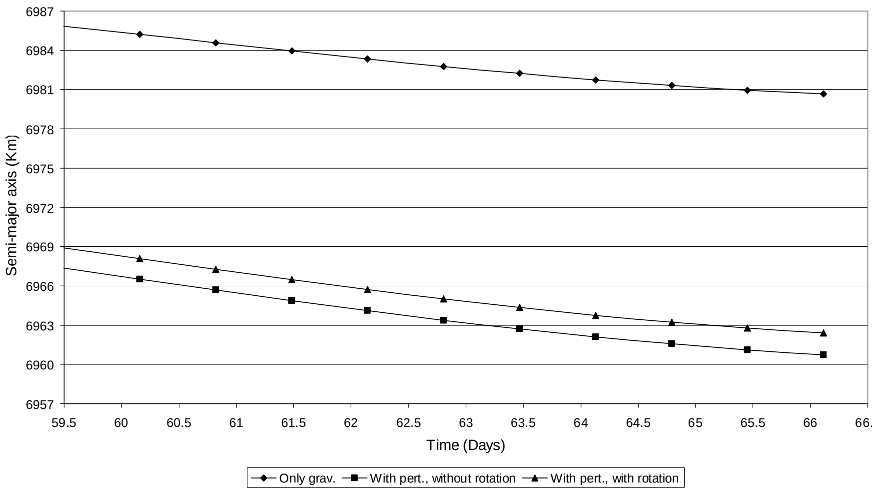


**Fig.(1-b): One thousand revolutions (model I).**

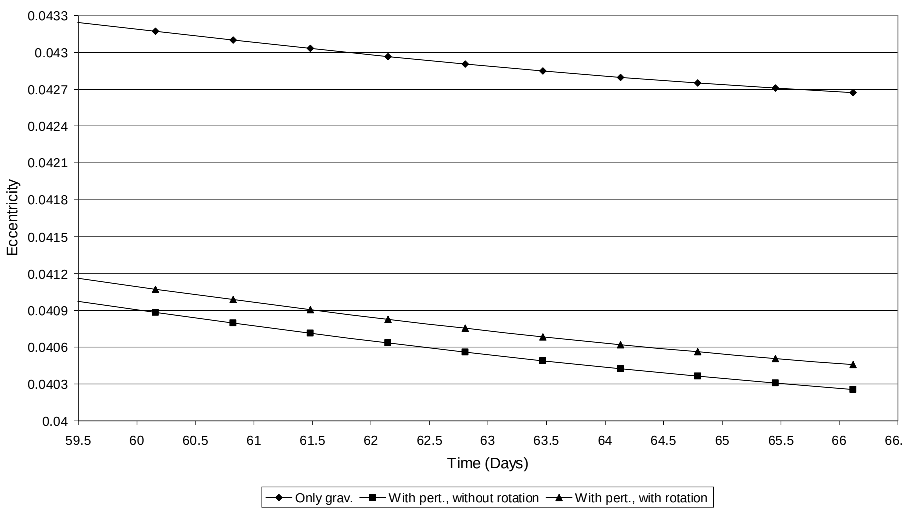

Fig.(1-c): One thousand revolutions (model I).

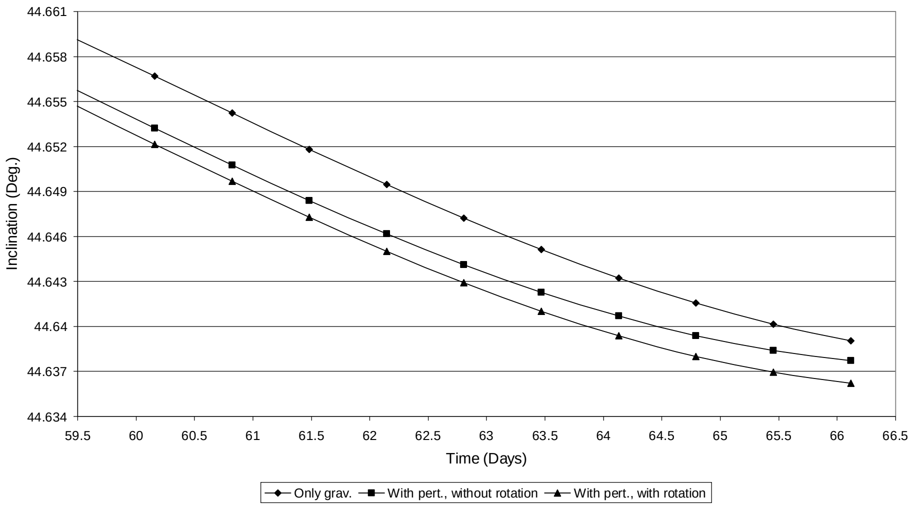


**Table (1): The values of bilinear relation correspond to their perturbation forces over one thousand revolutions.**

| Time (Days) | The bilinear relation (BI) | | |
|---|---|---|---|
| | Only grav. | With pert., without rotation | With pert., with rotation |
| 59.4970218402 | 8.788485593E-05 | 8.788477135E-05 | 8.788472951E-05 |
| 60.1588352089 | 8.788504965E-05 | 8.788500509E-05 | 8.788493778E-05 |
| 60.8206485775 | 8.788524246E-05 | 8.788515152E-05 | 8.788514970E-05 |
| 61.4824619461 | 8.788536070E-05 | 8.788541072E-05 | 8.788533160E-05 |
| 62.1442753148 | 8.788540526E-05 | 8.788562445E-05 | 8.788548803E-05 |
| 62.8060886834 | 8.788547348E-05 | 8.788570449E-05 | 8.788563355E-05 |
| 63.4679020520 | 8.788552168E-05 | 8.788576906E-05 | 8.788572086E-05 |
| 64.1297154207 | 8.788560353E-05 | 8.788576815E-05 | 8.788573905E-05 |
| 64.7915287893 | 8.788570449E-05 | 8.788568312E-05 | 8.788575315E-05 |
| 65.4533421579 | 8.788563355E-05 | 8.788573723E-05 | 8.788564310E-05 |
| 66.1151555266 | 8.788553214E-05 | 8.788573450E-05 | 8.788554578E-05 |

**Fig.(2-a): One thousand revolution (Model II).**

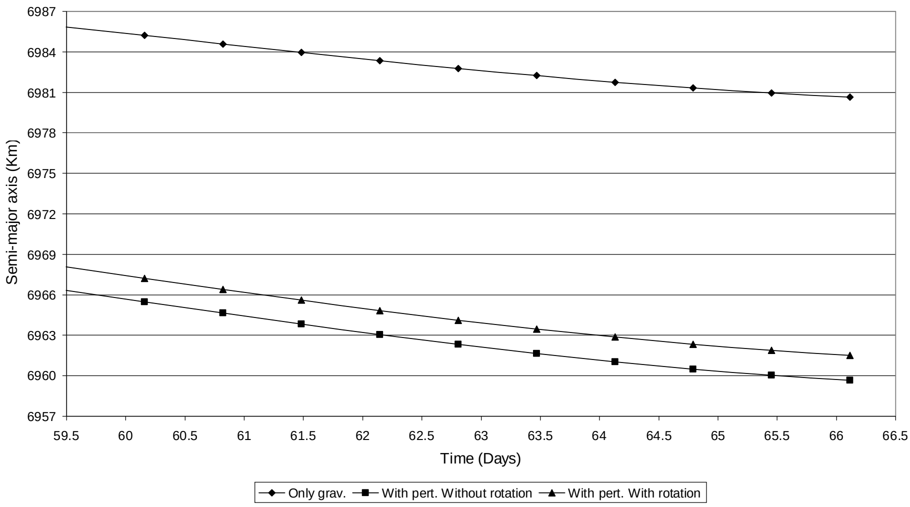


**Fig.(2-b): One thousand revolutions (Model II).**

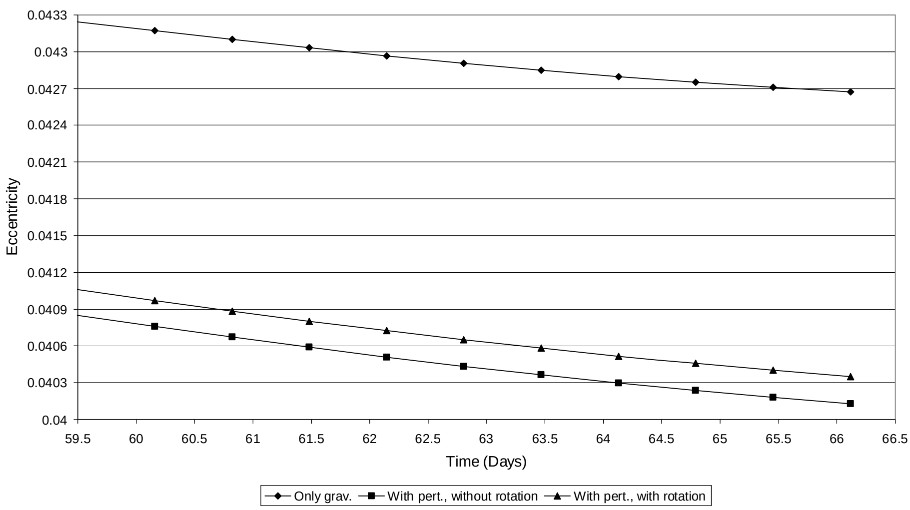

Fig.(2-c): One thousand revolutions (Model II).

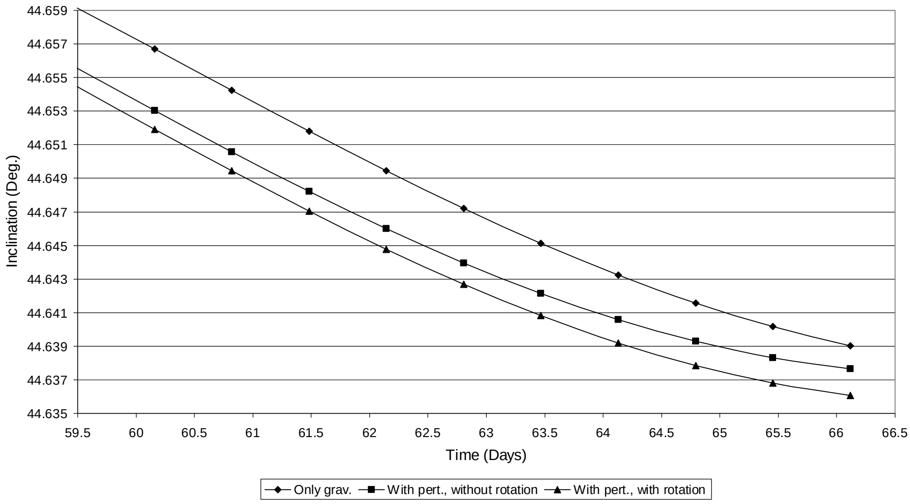


**Table (2): The values of bilinear relation correspond to their perturbation forces over one thousand revolutions.**

| Time (Days) | The bilinear relation (BI) | | |
|---|---|---|---|
| | Only grav. | With pert., without rotation | With pert., with rotation |
| 59.4970218402 | 8.788485593E-05 | 8.788434116E-05 | 8.788433297E-05 |
| 60.1588352089 | 8.788504965E-05 | 8.788452487E-05 | 8.788463765E-05 |
| 60.8206485775 | 8.788524246E-05 | 8.788472587E-05 | 8.788476953E-05 |
| 61.4824619461 | 8.788536070E-05 | 8.788489322E-05 | 8.788491141E-05 |
| 62.1442753148 | 8.788540526E-05 | 8.788495416E-05 | 8.788506511E-05 |
| 62.8060886834 | 8.788547348E-05 | 8.788504056E-05 | 8.788521973E-05 |
| 63.4679020520 | 8.788552168E-05 | 8.788515061E-05 | 8.788525247E-05 |
| 64.1297154207 | 8.788560353E-05 | 8.788514424E-05 | 8.788532796E-05 |
| 64.7915287893 | 8.788570449E-05 | 8.788511468E-05 | 8.788533387E-05 |
| 65.4533421579 | 8.788563355E-05 | 8.788506466E-05 | 8.788524792E-05 |
| 66.1151555266 | 8.788553214E-05 | 8.788498326E-05 | 8.788517380E-05 |

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