Open J. Math. Sci., Vol. 2(2018), No.1, pp. 39 - 55
Website: https://pisrt.org/psr-press/journals/oms/
ISSN: 2523-0212 (Online) 2616-4906 (Print)
http://dx.doi.org/10.30538/oms2018.0016

ON THE VISCOSITY RULE FOR COMMON FIXED POINTS OF TWO NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

EBENEZER BONYAH, MAQBOOL AHMAD[1], IFTIKHAR AHMAD

Abstract. In this paper, we establish viscosity rule for common fixed points of two nonexpansive mappings in the framework of CAT(0) spaces. The strong convergence theorems of the proposed technique is proved under certain assumptions imposed on the sequence of parameters. The results presented in this work extend and improve some recent announced in the literature.

Mathematics Subject Classification: 47J25, 47N20, 34G20, 65J15.
Key words and phrases: viscosity rule; CAT(0) space; nonexpansive mapping; variational inequality.

1. Introduction

The study of spaces of nonpositive curvature originated with the discovery of hyperbolic spaces, and flourished by pioneering works of J. Hadamard and E. Cartan in the first decades of the twentieth century. The idea of nonpositive curvature geodesic metric spaces could be traced back to the work of H. Busemann and A. D. Alexandrov in the 50's. Later on M. Gromov restated some features of global Riemannian geometry solely based on the so-called CAT(0) inequality (here the letters C, A and T stand for Cartan, Alexandrov and Toponogov, respectively). For through discussion of CAT(0) spaces and of fundamental role, they play in geometry, we refer the reader to Bridson and Haefliger [1].

As we know, iterative methods for finding fixed points of nonexpansive mappings have received vast investigations due to its extensive applications in a variety

Received 12 December 2017. Revised 26 March 2018.
[1] Corresponding Author

of applied areas of inverse problem, partial differential equations, image recovery, and signal processing; see [2, 3, 4, 5, 6, 7, 8, 9] and the references therein. One of the difficulties in carrying out results from Banach space to complete CAT(0) space setting lies in the heavy use of the linear structure of the Banach spaces. Berg and Nikolaev [10] introduce the notion of an inner product-like notion (quasi-linearization) in complete CAT(0) spaces to resolve these difficulties. Fixed-point theory in CAT(0) spaces was first studied by Kirk [11, 12]. He showed that every nonexpansive (singlevalued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed-point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed. In 2000, Moudaf's [13] introduced viscosity approximation methods as following

Theorem 1.1. [13] *Let C be a nonempty closed convex subset of the real Hilbert space X. Let T be a nonexpansive mapping of C into itself such that $Fix(T)$ is nonempty. Let f be a contraction of C into itself with coefficient $\theta \in [0,1)$. Pick any $x_0 \in [0,1)$, let $\{xn\}$ be a sequence generated by*

$$x_{n+1} = \frac{\gamma_n}{1+\gamma_n} f(x_n) + \frac{1}{1+\gamma_n} T(x_n), \quad n \geq 0$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(1) $\lim\limits_{n\to\infty} \gamma_n = 0$,

(2) $\sum\limits_{n=0}^{\infty} \gamma_n = \infty$,

(3) $\sum_{n=0}^{\infty} |\frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n}| = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the mapping T, which is also the unique solution of the variational inequality

$$\langle x - f(x), x - y\rangle \geq 0, \quad \forall\, y \in Fix(T),$$

in other words, x^* is the unique fixed point of the contraction $P_{Fix(T)}f$, that is $P_{Fix(T)}f(x^*) = x^*$.

Shi and Chen [14] studied the convergence theorems of the following Moudaf's viscosity iterations for a nonexpansive mapping in CAT(0) spaces.

$$x_{n+1} = tf(x_n) \oplus (1-t)T(x_n) \tag{1}$$

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(x_n) \tag{2}$$

They proved that $\{x_n\}$ defined by (1) and $\{x_n\}$ defined by (2) converged strongly to a fixed point of T in the framework of CAT(0) space. In 2017, Zhao *et al.* [15] applied viscosity approximation methods for the implicit midpoint rule for non-expansive mappings

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \forall n \geq 0.$$

Motivated and inspired by the idea of Naqvi *et al.* [16], in this paper, we establish viscosity rule for common fixed points of two nonexpansive mappings in the framework of CAT(0) spaces

$$x_{n+1} = \alpha_n f(x_n) \oplus \beta_n S\left(\frac{x_{n+1} \oplus x_n}{2}\right) + \gamma_n T\left(\frac{x_{n+1} \oplus x_n}{2}\right) \tag{3}$$

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2,and x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2} d(x_i, x_j) = d(x_i, x_j)$for $i, j = 1, 2, 3$.
A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.
Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) = d_{\mathbb{E}^2}(\overline{x}, \overline{y}) \tag{4}$$

Let $x, y \in X$ and by the Lemma 2.1(iv) of [17] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1 - t)d(x, y). \tag{5}$$

From now on, we will use the notation $(1 - t)x \oplus ty$ for the unique fixed point z satisfying the above equation.
We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.1. [17] *Let X be a CAT(0) spaces.*

- *For any $x, y, z \in X$ and $t \in [0, 1]$,*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z) \tag{6}$$

- *For any $x, y, z \in X$ and $t \in [0, 1]$,*

$$d^2((1 - t)x \oplus ty, z) \leq (1 - t)^2 d(x, z) + td^2(y, z) - t(1 - t)d^2(x, y) \tag{7}$$

Complete CAT(0) spaces are often called Hadamard spaces (see [1]). If x, y_1, y_2 are points of a CAT(0) spaces and y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denoted by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{8}$$

This inequality is the (CN) inequality of Bruhat and Tits [18]. In fact, a geodesic metric space is a CAT(0) space if and only if it satisfes the (CN) inequality (cf. [1], page 163).

Definition 2.2. Let X be a CAT(0) space and $T : X \to X$ be a mapping. Then T is called nonexpensive if

$$d(T(x), T(y)) \leq d(x, y), \quad x, y \in C.$$

Definition 2.3. Let X be a CAT(0) space and $T : X \to X$ be a mapping. Then T is called contraction if

$$d(T(x), T(y)) \leq \theta d(x, y), \quad x, y \in C \;\; \theta \in [0, 1).$$

Berg and Nikolaev [10] introduce the concept of quasilinearization as follow: Let us denote the pair $(a, b) \in X \times X$ by the $\overrightarrow{ab}$ and call it a vector. Then, quasilinearization is defined as a map

$$\langle ., . \rangle : (X \times X) \times (X \times X) \longrightarrow \mathbb{R}$$

defined as

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)). \tag{9}$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(a, c)$$

for all $a, b, c, d \in X$. It is well-known [10] that a geodesically connected metric space is a CAT(0) space of and only if it satisfy the Cauchy-Schwarz inequality. Let C be a non-empty closed convex subset of a complete CAT(0) space X. The metric projection $P_c : X \to C$ is defined by

$$u = P_c(x) \Longleftrightarrow \inf\{d(y, x) : y \in C\}, \quad \forall x \in X.$$

Definition 2.4. Let $P_c : X \to C$ is called the metric projection if for every $x \in X$ there exist a unique nearest point in C, denoted by $P_c x$, such that

$$d(x, P_c x) \leq d(x, y), \quad y \in C.$$

The following theorem gives you the conditions for a projection mapping to be non-expensive.

Theorem 2.5. *Let C be a non-empty closed convex subset of a real CAT(0) space X and $P_c : X \to X$ a metric projection. Then*

(1) $d(P_c x, P_c y) \leq \langle \overrightarrow{xy}, \overrightarrow{P_c x P_c y} \rangle$ *for all* $x, y \in X$,
(2) P_c *is non-expensive mapping , that is,* $d(x, p_c x) \leq d(x, y)$ *for all* $y \in C$,
(3) $\langle \overrightarrow{xP_c x}, \overrightarrow{yP_c y} \rangle \leq 0$ *for all* $x \in X$ *and* $y \in C$.

Further if, in addition, C is bounded, then $F(T)$ is nonempty. The following Lemmas are very useful for proving our main results:

Lemma 2.6. *(The demiclosedness principle) Let C be a nonempty closed convex subset of the real CAT(0) space X and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \text{ and } (I - T)x_n \to 0.$$

Then $x^* = Tx^*$. (Here $\to$ (respectively $\rightharpoonup$) denotes strong (respectively weak) convergence.)

Moreover, the following result gives the conditions for the convergence of a non-negative real sequences.

Lemma 2.7. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \beta_n)a_n + \delta_n, \forall n \geq 0$, where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence with*

(1) $\sum_{n=0}^{\infty} \beta_n = \infty$,
(2) $\lim_{n\to\infty} \sup \frac{\delta_n}{\beta_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$,

Then $\lim\limits_{n\to\infty} a_n \to 0$.

3. Main Result

Theorem 3.1. *Let C be a nonempty closed convex subset of the real CAT(0) spaces X. Let $S : C \to C$ and $T : C \to C$ be two nonexpansive mappings with $U := F(T) \cap F(S) \neq \phi$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus \beta_n S\left(\frac{x_{n+1} \oplus x_n}{2}\right) + \gamma_n T\left(\frac{x_{n+1} \oplus x_n}{2}\right),$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$, satisfying the following conditions:

(1) $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\lim\limits_{n\to\infty} \gamma_n = 1$;
(2) $\sum\limits_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum\limits_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(3) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;
(4) $\lim\limits_{n\to\infty} T\left(\frac{x_{n+1} \oplus x_n}{2}\right) - S\left(\frac{x_{n+1} \oplus x_n}{2}\right) = 0$ *and* $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \beta_n = 0$.

Then $\{x_n\}$ converges strongly to a common fixed point x^* of the nonexpansive mappings T and S which is also the unique solution of the variational inequality

$$\langle \overrightarrow{xf(x)}, \overrightarrow{yx} \rangle \quad \forall y \in U.$$

In other words, x^* is the unique fixed point of the contraction $P_U f$.

Proof. We will prove this theorem into the following five steps:

Step 1. First, we show that the sequence $\{x_n\}$ is bounded. Indeed, take $p \in U$ arbitrarily, we have

$$\begin{aligned}
d(x_{n+1}, p) &= d\left(\alpha_n f(x_n) \oplus \beta_n S\left(\frac{x_{n+1} \oplus x_n}{2}\right) \oplus \gamma_n T\left(\frac{x_{n+1} \oplus x_n}{2}\right), p\right) \\
&= d\left(\alpha_n f(x_n) \oplus \beta_n S\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right. \\
&\quad \left. + \gamma_n T\left(\frac{x_{n+1} \oplus x_n}{2}\right), (\alpha_n + \beta_n + \gamma_n)p\right)
\end{aligned}$$

$$\begin{aligned}
d(x_{n+1}, p) &\leq \alpha_n d(f(x_n), p) + \beta_n d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), p\right) \\
&\quad + \gamma_n d\left(T\left(\frac{x_{n+1} \oplus x_n}{2}\right), p\right) \\
&\leq \alpha_n d(f(x_n), f(p)) + \alpha_n d(f(p), p) + \beta_n d(\left(\frac{x_{n+1} \oplus x_n}{2}, p\right) \\
&\quad + \gamma_n d\left(\frac{x_{n+1} \oplus x_n}{2}, p\right) \\
&\leq \theta \alpha_n d(x_n, p) + \alpha_n d(f(p), p) + (\beta_n + \gamma_n) d\left(\frac{x_{n+1} \oplus x_n}{2}, p\right) \\
&= \theta \alpha_n d(x_n, p) + \alpha_n d(f(p), p) + (1 - \alpha_n) d\left(\frac{x_{n+1} \oplus x_n}{2}, p\right) \\
&\leq \theta \alpha_n d(x_n, p) + \alpha_n d(f(p), p) + \frac{1 - \alpha_n}{2} d(x_{n+1}, p) \\
&\quad + \frac{1 - \alpha_n}{2} d(x_n, p),
\end{aligned}$$

this is equivalent to

$$\left(1 - \frac{1 - \alpha_n}{2}\right) d(x_{n+1}, p) \leq \left(\frac{1 - \alpha_n}{2} + \alpha_n \theta\right) d(x_n, p) + \alpha_n d(f(p), p)$$

$\Rightarrow$

$$(1 + \alpha_n) d(x_{n+1}, p) \leq (1 - \alpha_n + 2\alpha_n \theta) d(x_n, p) + 2\alpha_n d(f(p), p)$$

$\Rightarrow$

$$\begin{aligned}
d(x_{n+1}, p) &\leq \frac{1 + \alpha_n - 2\alpha_n + 2\alpha_n \theta}{1 + \alpha_n} d(x_n, p) + \frac{2\alpha_n}{1 + \alpha_n} d(f(p), p) \\
&= \left(1 - \frac{2\alpha_n(1 - \theta)}{1 + \alpha_n}\right) d(x_n, p)
\end{aligned}$$

$$+\frac{2\alpha_n(1-\theta)}{1+\alpha_n}\left(\frac{1}{1-\theta}d(f(p),p)\right)$$

Thus,

$$d(x_{n+1},p) \leq \max\left\{d(x_n,p), \frac{1}{1-\theta}d(f(p),p)\right\}.$$

Similarly,

$$d(x_n,p) \leq \max\left\{d(x_{n-1},p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\}.$$

From this, we obtain,

$$\begin{aligned} d(x_{n+1},p) &\leq \max\left\{d(x_n,p), \frac{1}{1-\theta}d(f(p),p)\right\} \\ &\leq \max\left\{d(x_{n-1},p), \frac{1}{1-\theta}d(f(p),p)\right\} \\ &. \\ &. \\ &. \\ &. \\ &\leq \max\left\{d(x_0,p), \frac{1}{1-\theta}d(f(p),p)\right\}. \end{aligned}$$

Hence, we concluded that $\{x_n\}$ is a bounded sequence. Consequently, $\{f(x_n)\}$, $\{S(\frac{x_{n+1}\oplus x_n}{2})\}$ and $\{T(\frac{x_{n+1}\oplus x_n}{2})\}$ are bounded.

Step 2. Now, we prove that $\lim\limits_{n\to\infty} d(x_{n+1},x_n)=0$.

$$\begin{aligned} & d(x_{n+1},x_n) \\ = \; & d\left(\alpha_n f(x_n) \oplus \beta_n S\left(\frac{x_{n+1}\oplus x_n}{2}\right) + \gamma_n T\left(\frac{x_{n+1}\oplus x_n}{2}\right)\right. \\ & \left., \alpha_{n-1} f(x_{n-1}) \oplus \beta_{n-1} S\left(\frac{x_n\oplus x_{n-1}}{2}\right) + \gamma_{n-1} T\left(\frac{x_n\oplus x_{n-1}}{2}\right)\right) \\ \leq \; & \alpha_n d\big(f(x_n), f(x_{n-1})\big) + |\alpha_n-\alpha_{n-1}| d\left(f(x_{n-1}), T\left(\frac{x_n\oplus x_{n-1}}{2}\right)\right) \\ & +\beta_n d\left(S\left(\frac{x_{n+1}\oplus x_n}{2}\right), S\left(\frac{x_n\oplus x_{n-1}}{2}\right)\right) \\ & +|\beta_n-\beta_{n-1}| d\left(S\left(\frac{x_n\oplus x_{n-1}}{2}\right), T\left(\frac{x_n\oplus x_{n-1}}{2}\right)\right) \\ & +\gamma_n d\left(T\left(\frac{x_{n+1}\oplus x_n}{2}\right), T\left(\frac{x_n\oplus x_{n-1}}{2}\right)\right). \end{aligned}$$

Let M_2 be a number such that $M_2 \geq \max\left\{\sup\limits_{n\geq 0} d(S\left(\frac{x_{n+1}\oplus x_n}{2}\right), T\left(\frac{x_{n+1}\oplus x_n}{2}\right)), \sup\limits_{n\geq 0} d(f(x_n), T\left(\frac{x_{n+1}\oplus x_n}{2}\right))\right\}$. Thus, the above is equivalent to

$$\begin{aligned}
& \|x_{n+1}-x_n\| \\
\leq\; & \alpha_n\theta d(x_n,x_{n-1})+\beta_n d(\frac{x_{n+1},x_n}{2},\frac{x_n+x_{n-1}}{2}) \\
& +\gamma_n d\left(\frac{x_{n+1}+x_n}{2},\frac{x_n+x_{n-1}}{2}\right)+|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|M_2 \\
\leq\; & \alpha_n\theta d(x_n,x_{n-1})+\frac{\beta_n}{2}d(x_{n+1},x_n)+\frac{\beta_n}{2}d(x_n,x_{n-1})+\frac{\gamma_n}{2}d(x_{n+1},x_n) \\
& +\frac{\gamma_n}{2}d(x_n,x_{n-1})+|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|M_2 \\
=\; & \left(\alpha_n\theta+\frac{\beta_n}{2}+\frac{\gamma_n}{2}\right)d(x_n,x_{n-1})+\left(\frac{\beta_n}{2}+\frac{\gamma_n}{2}\right)d(x_{n+1},x_n) \\
& +(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)M_2 \\
=\; & \left(\alpha_n\theta+\frac{1-\alpha_n}{2}\right)d(x_n,x_{n-1})+\frac{1-\alpha_n}{2}d(x_{n+1},x_n) \\
& +(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)M_2.
\end{aligned}$$

Combining the common terms from left and right hand sides, we get,

$$\begin{aligned}
\left(1-\frac{1-\alpha_n}{2}\right)d(x_{n+1},x_n) \quad\leq\quad & \left(\alpha_n\theta+\frac{1-\alpha_n}{2}\right)d(x_n,x_{n-1}) \\
& +(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)M_2
\end{aligned}$$

This implies that

$$\begin{aligned}
& d(x_{n+1},x_n) \\
\leq\; & \frac{1+\alpha_n-2\alpha_n+2\alpha_n\theta}{1+\alpha_n}d(x_n,x_{n-1})+\frac{2(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)}{1+\alpha_n}M_2 \\
=\; & \left(1-\frac{2\alpha_n(1-\theta)}{1+\alpha_n}\right)d(x_n,x_{n-1})+\frac{2(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)}{1+\alpha_n}M_2
\end{aligned}$$

Note that $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$, $\sum\limits_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ and $\sum\limits_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty$. Using Lemma 2.7, we have $d(x_{n+1},x_n)\to 0$ as $n\to\infty$.

Step 3. Now, we will show that $\lim\limits_{n\to\infty}d(x_n,Sx_n)=0$ and $\lim\limits_{n\to\infty}d(x_n,Tx_n)=0$. Consider

$$\begin{aligned}
& d(x_n,S(x_n)) \\
\leq\; & d(x_n,x_{n+1})+d\left(x_{n+1},T\left(\frac{x_{n+1}\oplus x_n}{2}\right)\right)+d\left(T\left(\frac{x_{n+1}+x_n}{2}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& , S\left(\frac{x_{n+1} \oplus x_n}{2}\right)\Bigg) + d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), S(x_n)\right) \\
\leq\ & d(x_n, x_{n+1}) + d\Bigg(\alpha_n f(x_n) \oplus \beta_n S\left(\frac{x_{n+1} \oplus x_n}{2}\right) \oplus \gamma_n T\left(\frac{x_{n+1} \oplus x_n}{2}\right) \\
& , T\left(\frac{x_{n+1} \oplus x_n}{2}\right)\Bigg) + d(\frac{x_{n+1} \oplus x_n}{2}, x_n) \\
& + d\left(T\left(\frac{x_{n+1} \oplus x_n}{2}\right), S\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) \\
\leq\ & d(x_n, x_{n+1}) + \alpha_n d\left(f(x_n), T\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) \\
& + \beta_n d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), T\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) + \frac{1}{2} d(x_{n+1}, x_n) \\
& + d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), T\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) \\
\leq\ & \frac{3}{2} d(x_n, x_{n+1}) + \alpha_n d\left(f(x_n), T\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) \\
& + (1 + \beta_n) d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), T\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right)
\end{aligned}
$$

Since, $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} d\left(T\left(\frac{x_{n+1}\oplus x_n}{2}\right), S\left(\frac{x_{n+1}\oplus x_n}{2}\right)\right) = 0$ and $\lim_{n\to\infty} d(x_{n+1}, x_n) \to 0$, we get $d(x_n, S(x_n)) \to 0$ as $n \to \infty$.

Moreover, we have

$$
\begin{aligned}
d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), x_n\right) &\leq d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), S(x_n)\right) + d\left(S(x_n), x_n\right) \\
&\leq d\left(\frac{x_{n+1} \oplus x_n}{2}, x_n\right) + d\left(S(x_n), x_n\right) \\
&= \frac{1}{2} d(x_{n+1}, x_n) + d(S(x_n), x_n) \\
&\to 0 \qquad \text{as } (n \to \infty)
\end{aligned}
$$

Now, consider

$$
\begin{aligned}
& d(x_n, T(x_n)) \\
\leq\ & d(x_n, x_{n+1}) + d\left(x_{n+1}, S\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) + d\left(T\left(\frac{x_{n+1} \oplus x_n}{2}\right), T(x_n)\right) \\
& + d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), T\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) \\
\leq\ & d(x_n, x_{n+1}) + d\left(T\left(\frac{x_{n+1} \oplus x_n}{2}\right), S\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) +
\end{aligned}
$$

$$\begin{aligned}
&\left(\alpha_n f(x_n) \oplus \beta_n S\left(\frac{x_{n+1} \oplus x_n}{2}\right) \oplus \gamma_n T\left(\frac{x_{n+1} \oplus x_n}{2}\right), S\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) \\
&+d\left(\frac{x_{n+1} \oplus x_n}{2}, x_n\right) \\
\leq\ & d(x_n, x_{n+1}) + \alpha_n d\left(f(x_n), T\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) + \frac{1}{2} d(x_{n+1}, x_n) \\
&+\gamma_n d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), T\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) \\
&+d(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), T\left(\frac{x_{n+1} \oplus x_n}{2}\right)) \\
\leq\ & \frac{3}{2} d(x_n, x_{n+1}) + \alpha_n d\left(f(x_n), T\left(\frac{x_{n+1} + x_n}{2}\right)\right) \\
&+(1+\gamma_n) d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), T\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right)
\end{aligned}$$

Since, $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} d\left(T\left(\frac{x_{n+1} \oplus x_n}{2}\right), S\left(\frac{x_{n+1} \oplus x_n}{2}\right)\right) = 0$ and $\lim_{n\to\infty} d(x_{n+1}, x_n) \to 0$, we get $d(x_n, Tx_n) \to 0$ as $n \to \infty$.
Also,

$$\begin{aligned}
d\left(T\left(\frac{x_{n+1} \oplus x_n}{2}\right), x_n\right) &\leq d\left(T\left(\frac{x_{n+1} \oplus x_n}{2}\right), T(x_n)\right) + d(T(x_n), x_n) \\
&\leq d\left(\frac{x_{n+1} \oplus x_n}{2}, x_n\right) + d(T(x_n), x_n) \\
&= \frac{1}{2} d(x_{n+1}, x_n) + d(T(x_n), x_n) \\
&\to 0 \qquad (\text{ as } n \to \infty)
\end{aligned}$$

Step 4. In this step, we will show that $\limsup_{n\to\infty} \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_n} \rangle \leq 0$, where, $x^* = P_U f(x^*)$.

Indeed, we take a subsequence, $\{x_{n_i}\}$ of $\{x_n\}$, which converges weakly to a fixed point $p \in U = F(T) \cap F(S)$. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup p$. From $\lim_{n\to\infty} d(x_n, S(x_n)) = 0$, $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ and Lemma 2.6, we have $p = S(p)$ and $p = T(p)$. This together with the property of the metric projection implies that

$$\limsup_{n\to\infty} \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_n} \rangle = \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* p} \rangle \leq 0$$

Step 5. Finally, we show that $x_n \to x^*$ as $n \to \infty$. Again, take $x^* \in U$ to be the unique fixed point of the contraction $P_U f$. Consider

$$
\begin{aligned}
& d^2(x_{n+1}, x_n) \\
= \; & d^2\left(\alpha_n f(x_n) \oplus \beta_n S\left(\frac{x_{n+1} \oplus x_n}{2}\right) + \gamma_n T\left(\frac{x_{n+1} \oplus x_n}{2}\right), x^*\right) \\
= \; & d^2\left(\alpha_n f(x_n) \oplus \beta_n S\left(\frac{x_{n+1} \oplus x_n}{2}\right) + \gamma_n T\left(\frac{x_{n+1} \oplus x_n}{2}\right), (\alpha_n + \beta_n + \gamma_n) x^*\right) \\
= \; & \alpha_n^2 d^2(f(x_n), x^*) + \beta_n^2 d^2\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), x^*\right) \\
& + \gamma_n^2 d^2\left(T\left(\frac{x_{n+1} \oplus x_n}{2}\right), x^*\right) \\
& + 2\alpha_n\beta_n \left\langle \overrightarrow{f(x_n)x^*}, \overrightarrow{S\left(\frac{x_{n+1} \oplus x_n}{2}\right) x^*} \right\rangle \\
& + 2\alpha_n\gamma_n \left\langle \overrightarrow{f(x_n)x^*}, \overrightarrow{T\left(\frac{x_{n+1} \oplus x_n}{2}\right) x^*} \right\rangle \\
& + 2\beta_n\gamma_n \left\langle \overrightarrow{S\left(\frac{x_{n+1} \oplus x_n}{2}\right) x^*}, \overrightarrow{T\left(\frac{x_{n+1} \oplus x_n}{2}\right) x^*} \right\rangle \\
\leq \; & \alpha_n^2 d^2(f(x_n), x^*) + \beta_n^2 d^2\left(\frac{x_{n+1} \oplus x_n}{2}, x^*\right) + \gamma_n^2 d^2\left(\frac{x_{n+1} \oplus x_n}{2}, x^*\right) \\
& + 2\alpha_n\beta_n \left\langle \overrightarrow{f(x_n)f(x^*)}, \overrightarrow{S\left(\frac{x_{n+1} \oplus x_n}{2}\right) x^*} \right\rangle \\
& + 2\alpha_n\beta_n \left\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{S\left(\frac{x_{n+1} \oplus x_n}{2}\right) x^*} \right\rangle \\
& + 2\alpha_n\gamma_n \left\langle \overrightarrow{f(x_n)f(x^*)}, \overrightarrow{T\left(\frac{x_{n+1} \oplus x_n}{2}\right) x^*} \right\rangle \\
& + 2\alpha_n\gamma_n \left\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T\left(\frac{x_{n+1} \oplus x_n}{2}\right) x^*} \right\rangle \\
& + 2\beta_n\gamma_n \left\langle \overrightarrow{S\left(\frac{x_{n+1} \oplus x_n}{2}\right) x^*}, \overrightarrow{T\left(\frac{x_{n+1} \oplus x_n}{2}\right) x^*} \right\rangle \\
\leq \; & (\beta_n^2 + \gamma_n^2) d^2\left(\frac{x_{n+1} \oplus x_n}{2}, x^*\right) + 2\alpha_n\beta_n d(f(x_n), f(x^*)) \\
& \times d\left(S\left(\frac{x_{n+1} \oplus x_n}{2}\right), x^*\right) + 2\alpha_n\gamma_n d(f(x_n), f(x^*)).d\left(T\left(\frac{x_{n+1} \oplus x_n}{2}\right), x^*\right)
\end{aligned}
$$

$$+2\beta_n\gamma_n d\left(S\left(\frac{x_{n+1}\oplus x_n}{2}\right),x^*\right).d\left(T\left(\frac{x_{n+1}\oplus x_n}{2}\right),x^*\right)+K_n$$

where

$$\begin{aligned} K_n &= \alpha_n^2 d^2(f(x_n),x^*)+2\alpha_n\beta_n\left\langle \overrightarrow{f(x^*)x^*},\overrightarrow{S\left(\frac{x_{n+1}\oplus x_n}{2}\right)x^*}\right\rangle \\ &\quad +2\alpha_n\gamma_n\left\langle \overrightarrow{f(x^*)x^*},\overrightarrow{T\left(\frac{x_{n+1}\oplus x_n}{2}\right)x^*}\right\rangle \end{aligned}$$

This, implies that

$$\begin{aligned} & d^2(x_{n+1},x_n) \\ \leq\ & (\beta_n^2+\gamma_n^2)d^2\left(\frac{x_{n+1}\oplus x_n}{2},x^*\right)+2\alpha_n\beta_n\theta d(x_n,x^*).d\left(\frac{x_{n+1}\oplus x_n}{2},x^*\right) \\ & +2\alpha_n\gamma_n\theta d(x_n,x^*).d\left(\frac{x_{n+1}\oplus x_n}{2},x^*\right) \\ & +2\beta_n\gamma_n d\left(\frac{x_{n+1}\oplus x_n}{2},x^*\right).d\left(\frac{x_{n+1}\oplus x_n}{2},x^*\right)+K_n \\ =\ & (\beta_n^2+\gamma_n^2+2\beta_n\gamma_n)d^2\left(\frac{x_{n+1}\oplus x_n}{2},x^*\right) \\ & +2\alpha_n\theta(\beta_n+\gamma_n)d(x_n,x^*).d\left(\frac{x_{n+1}\oplus x_n}{2},x^*\right)+K_n \\ =\ & (\beta_n+\gamma_n)^2d^2\left(\frac{x_{n+1}\oplus x_n}{2},x^*\right)) \\ & +2\alpha_n\theta(\beta_n+\gamma_n)d(x_n,x^*).d\left(\frac{x_{n+1}\oplus x_n}{2},x^*\right)+K_n \\ =\ & (1-\alpha_n)^2d^2(\frac{x_{n+1}\oplus x_n}{2},x^*) \\ & +2\alpha_n\theta(1-\alpha_n)d(x_n,x^*).d\left(\frac{x_{n+1}\oplus x_n}{2},x^*\right)+K_n \end{aligned}$$

The above calculation shows that

$$\begin{aligned} 0 &\leq 2\alpha_n\theta(1-\alpha_n)d(x_n,x^*).d(\frac{x_{n+1}\oplus x_n}{2}-x^*) \\ &\quad +(1-\alpha_n)^2d^2\left(\frac{x_{n+1}\oplus x_n}{2}-x^*\right)-d(x_{n+1},x^*)^2+K_n, \end{aligned}$$

which is a quadratic inequality in $d\left(\frac{x_{n+1}\oplus x_n}{2}, x^*\right)$. Solving the above inequality for $d\left(\frac{x_{n+1}\oplus x_n}{2}, x^*\right)$, we have

$$\begin{aligned}
& d\left(\frac{x_{n+1}+x_n}{2}, x^*\right) \\
\geq\ & \frac{-2\theta\alpha_n(1-\alpha_n)d(x_n,x^*)}{2(1-\alpha_n)^2} \\
& + \frac{\sqrt{4\theta^2\alpha_n^2(1-\alpha_n)^2 d^2(x_n,x^*) - 4(1-\alpha_n)^2(K_n - d^2(x_{n+1},x^*))}}{2(1-\alpha_n)^2} \\
=\ & \frac{-\theta\alpha_n d(x_n,x^*) + \sqrt{\theta^2\alpha_n^2 d^2(x_n,x^*) - K_n + d(x_{n+1},x^*)}}{1-\alpha_n}.
\end{aligned}$$

This will give

$$\begin{aligned}
& \frac{1}{2}\left(d(x_{n+1},x^*) + d(x_n,x^*)\right) \\
\geq\ & \frac{-\theta\alpha_n d(x_n,x^*) + \sqrt{\theta^2\alpha_n^2 d^2(x_n,x^*) - K_n + d^2(x_{n+1},x^*)}}{1-\alpha_n}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
& \frac{1}{2}\left((1-\alpha_n)d(x_{n+1},x^*) + (1+(2\theta-1)\alpha_n)d(x_n,x^*)\right) \\
\geq\ & \sqrt{\theta^2\alpha_n^2 d^2(x_n,x^*) - K_n + d^2(x_{n+1},x^*)}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
& \frac{1}{4}\left((1-\alpha_n)d(x_{n+1},x^*) + (1+(2\theta-1)\alpha_n)d(x_n,x^*)\right)^2 \\
\geq\ & \theta^2\alpha_n^2 d^2(x_n,x^*) - K_n + d^2(x_{n+1},x^*),
\end{aligned}$$

which is reduced to

$$\begin{aligned}
& \frac{1}{4}(1-\alpha_n)^2 d^2(x_{n+1},x^*) + \frac{1}{4}(1+(2\theta-1)\alpha_n)^2 d^2(x_n,x^*) \\
+\ & \frac{1}{2}(1-\alpha_n)(1+(2\theta-1)\alpha_n)d(x_{n+1},x^*).d(x_n,x^*) \\
\geq\ & \theta^2\alpha_n^2 d(x_n,x^*) - K_n + d^2(x_{n+1}-x^*).
\end{aligned}$$

This inequality is further reduced by using the elementary inequality

$$2d(x_{n+1},x^*)d(x_n,x^*) \quad \leq \quad d^2(x_{n+1},x^*) + d^2(x_n,x^*)$$

to the following inequality

$$\begin{aligned}
& \frac{1}{4}(1-\alpha_n)^2 d^2(x_{n+1},x^*) + \frac{1}{4}(1+(2\theta-1)\alpha_n)^2 d^2(x_n,x^*) \\
+\ & \frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)(d^2(x_{n+1},x^*) + d^2(x_n,x^*)) \\
\geq\ & \theta^2\alpha_n^2 d^2(x_n-x^*) - K_n + d^2(x_{n+1},x^*)
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left(1-\frac{1}{4}(1-\alpha_n)^2-\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)\right)d^2(x_{n+1},x^*)\\
\leq\ &\left(\frac{1}{4}(1+(2\theta-1)\alpha_n)^2+\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)-\theta^2\alpha_n^2\right)d^2(x_n,x^*)\\
&+K_n
\end{aligned}$$

or

$$\begin{aligned}
d^2(x_{n+1},x^*) \leq\ & \frac{\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)-\theta^2\alpha_n^2}{1-\frac{1}{4}(1-\alpha_n)^2-\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)}d(x_n,x^*)\\
&+\frac{\frac{1}{4}(1+(2\theta-1)\alpha_n)^2}{1-\frac{1}{4}(1-\alpha_n)^2-\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)}+K_n',\quad(10)
\end{aligned}$$

where

$$K_n' = \frac{K_n}{1-\frac{1}{4}(1-\alpha_n)^2-\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)}$$

Note that

$$\begin{aligned}
&1-\frac{1}{4}(1-\alpha_n)^2-\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)\\
=\ &1-\frac{1}{4}(1-\alpha_n)(1-\alpha_n+1+(2\theta-1)\alpha_n)\\
=\ &1-\frac{1}{4}(1-\alpha_n)(1-\alpha_n+1+2\theta\alpha_n-\alpha_n)\\
=\ &1-\frac{1}{4}(1-\alpha_n)(2-2\alpha_n+2\theta\alpha_n)\\
=\ &1-\frac{1}{2}(1-\alpha_n)(1-\alpha_n+\theta\alpha_n)\\
=\ &1-\frac{1}{2}(1-\alpha_n)(1-\alpha_n(1-\theta))
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{4}(1+(2\theta-1)\alpha_n)^2+\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)-\theta^2\alpha_n^2\\
=\ &\frac{1}{4}(1+(2\theta-1)\alpha_n)(1+(2\theta-1)\alpha_n+1-\alpha_n)-\theta^2\alpha_n^2\\
=\ &\frac{1}{4}(1+(2\theta-1)\alpha_n)(2+2\theta\alpha_n-2\alpha_n)-\theta^2\alpha_n^2\\
=\ &\frac{1}{2}(1+(2\theta-1)\alpha_n)(1+\theta\alpha_n-\alpha_n)-\theta^2\alpha_n^2\\
=\ &\frac{1}{2}(1+(2\theta-1)\alpha_n)(1-(1-\theta)\alpha_n)-\theta^2\alpha_n^2.
\end{aligned}$$

Now from (10),

$$\begin{aligned} & d^2(x_{n+1}, x^*) \\ \leq \quad & \frac{\frac{1}{2}(1+(2\theta-1)\alpha_n)(1-(1-\theta)\alpha_n)-\theta^2\alpha_n^2}{1-\frac{1}{2}(1-\alpha_n)(1-\alpha_n(1-\theta))} d^2(x_n, x^*) + K'_n. \end{aligned} \tag{11}$$

Consider the following function, for $t > 0$.

$$g(t) := \frac{1}{t}\left\{1 - \frac{\frac{1}{2}(1+(2\theta-1)t)(1-(1-\theta)t)-\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\}$$

$$\begin{aligned} g(t) \quad &= \quad \frac{1}{t}\left\{\frac{1-\frac{1}{2}(1-t)(1-t(1-\theta))-\frac{1}{2}(1+(2\theta-1)t)(1-(1-\theta)t)+\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\} \\ &= \quad \frac{1}{t}\left\{\frac{1-\frac{1}{2}(1-t(1-\theta))(1-t+1+2\theta t-t)+\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\} \\ &= \quad \frac{1}{t}\left\{\frac{1-\frac{1}{2}(1-t(1-\theta))(2-2t+2\theta t)+\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\} \\ &= \quad \frac{1}{t}\left\{\frac{1-(1-t+\theta t))(1-t+\theta t))+\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\} \\ &= \quad \frac{1}{t}\left\{\frac{1-(1+t^2+\theta^2t^2-2t-2\theta t^2+2\theta t)+\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\} \\ &= \quad \frac{1}{t}\left\{\frac{1-1-t^2-\theta^2t^2+2t+2\theta t^2-2\theta t+\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\} \\ &= \quad \frac{-t+2+2\theta t-2\theta}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}. \end{aligned}$$

By applying limit $t \to 0$, we have

$$\lim_{t\to 0} g(t) = 4(1-\theta) > 0.$$

Let $\delta > 0$ be such that for all $0 < t < \delta$, $g(t) > \epsilon := 4(1-\theta) > 0$. This is equivalent to

$$\frac{1}{t}\left\{1 - \frac{\frac{1}{2}(1+(2\theta-1)t)(1-(1-\theta)t)-\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\} \quad > \quad \epsilon$$

This implies,

$$1 - t\epsilon > \frac{\frac{1}{2}(1+(2\theta-1)t)(1-(1-\theta)t)-\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}.$$

Since $\alpha_n \to 0$ as $n \to \infty$, there exist some integer N, such that $\alpha_n < \delta, \forall\, n \geq N$. From (11), we have

$$d^2(x_{n+1}, x^*) \leq (1-\alpha_n\epsilon)d(x_n, x^*) + K'_n$$

On the other hand, we have

$$\begin{aligned}\limsup_{n\to\infty}\frac{K_n}{\alpha_n} &= \limsup_{n\to\infty}\Bigg\{2\beta_n\left\langle\overrightarrow{f(x^*)x^*},\overrightarrow{S\left(\frac{x_{n+1}\oplus x_n}{2}\right)x^*}\right\rangle\\ &\quad +2\gamma_n\left\langle\overrightarrow{f(x^*)x^*},\overrightarrow{T\left(\frac{x_{n+1}+x_n}{2}\right)x^*}\right\rangle\\ &\quad +\alpha_n d^2(f(x_n),x^*)\Bigg\}\\ &\leq 0\end{aligned}$$

The above inequality implies that

$$\limsup_{n\to\infty}\frac{K'_n}{\alpha_n} \leq 0.$$

From the above two inequalities and Lemma 2.6 we have

$$\lim_{n\to\infty} d^2(x_{n+1},x^*)=0,$$

which implies that $x_n \to x^*$ as $n\to\infty$. This completes the proof. □

Competing Interests

The authors do not have any competing interests in the manuscript.

References

1. Bridson, M. R., & Haefliger, A. (1999). Metric spaces of non-positive curvature, volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].
2. Alghamdi, M. A., Alghamdi, M. A., Shahzad, N., & Xu, H. K. (2014). The implicit midpoint rule for nonexpansive mappings. *Fixed Point Theory and Applications,* 2014(1), 96. https://doi.org/10.1186/1687-1812-2014-96
3. Attouch, H. (1996). Viscosity solutions of minimization problems. *SIAM J. Optim.,6*(3), 769806. (38 pages).https://doi.org/10.1137/S1052623493259616
4. Auzinger, W., & Frank, R. (1989). Asymptotic error expansions for stiff equations: an analysis for the implicit midpoint and trapezoidal rules in the strongly stiff case. *Numerische Mathematik,* 56(5), 469-499. https://doi.org/10.1007/BF01396649
5. Petrusel, A., & Yao, J. C. (2008). Viscosity approximation to common fixed points of families of nonexpansive mappings with generalized contractions mappings. *Nonlinear Analysis: Theory, Methods & Applications*, 69(4), 1100-1111. https://doi.org/10.1016/j.na.2007.06.016
6. Shimoji, K., & Takahashi, W. (2001). Strong convergence to common fixed points of infinite nonexpansive mappings and applications. *Taiwanese Journal of Mathematics,* 5(2), 387-404. doi:10.11650/twjm/1500407345
7. Wu, D., Chang, S. S., & Yuan, G. X. (2005). Approximation of common fixed points for a family of finite nonexpansive mappings in Banach space. *Nonlinear Analysis: Theory, Methods & Applications*, 63(5-7), 987-999.

8. Xu, H. K. (2004). Viscosity approximation methods for nonexpansive mappings. *Journal of Mathematical Analysis and Applications*, 298(1), 279-291. https://doi.org/10.1016/j.jmaa.2004.04.059
9. Yao, Y., & Shahzad, N. (2011). New methods with perturbations for non-expansive mappings in Hilbert spaces. *Fixed Point Theory and Applications*, 2011(1), 79. https://doi.org/10.1186/1687-1812-2011-79
10. Berg, I. D., & Nikolaev, I. G. (2008). Quasilinearization and curvature of Aleksandrov spaces. *Geometriae Dedicata*, 133(1), 195-218. https://doi.org/10.1007/s10711-008-9243-3
11. Kirk, W. A. (2003, February). Geodesic geometry and fixed point theory. *In Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003)* (Vol. 64, pp. 195-225).
12. Kirk, W. A. (2004). Geodesic geometry and fixed point theory II. *In International Conference on Fixed Point Theory and Applications* (pp. 113-142).
13. Moudafi, A. (2000). Viscosity approximation methods for fixed-points problems. *Journal of Mathematical Analysis and Applications,* 241(1), 46-55. https://doi.org/10.1006/jmaa.1999.6615
14. Shi, L. Y., & Chen, R. D. (2012). Strong convergence of viscosity approximation methods for nonexpansive mappings in CAT (0) spaces. *Journal of Applied Mathematics*, Volume 2012, Article ID 421050, 11 pages. http://dx.doi.org/10.1155/2012/421050.
15. Zhao, L. C., Chang, S. S., Wang, L., & Wang, G. (2017). Viscosity approximation methods for the implicit midpoint rule of nonexpansive mappings in CAT (0) Spaces. *J. Nonlinear Sci. Appl., 10*, 386394.
16. Naqvi, S. F. A., & Khan, M. S. (2017). On the viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces. *Open J. Math. Sci.,* Volume 1(2017), 111-125. http://dx.doi.org/10.30538/oms2017.0011
17. Dhompongsa, S., & Panyanak, B. (2008). On $\triangle$-convergence theorems in CAT (0) spaces. *Computers & Mathematics with Applications*, 56(10), 2572-2579. https://doi.org/10.1016/j.camwa.2008.05.036
18. Bruhat, F., & Tits, J. (1972). Groupes rductifs sur un corps local. *Publications Mathmatiques de l'Institut des Hautes tudes Scientifiques,* 41(1), 5-251.

Ebenezer Bonyah

Information Technology Education, University of Education, Winneba(Kumasi Campus) Ghana.
e-mail: ebbonya@gmail.com

Maqbool Ahmad

Department of Mathematics and Statistics, University of Lahore, Lahore Pakistan.
e-mail: maqboolchaudhri@gmail.com

Iftikhar Ahmad

Department of Mathematics and Statistics, University of Lahore, Lahore Pakistan.
e-mail: iftikharcheema1122@gmail.com