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Rank of the Subsemigroup of the Semigroup of Finite Full Contraction Maps Generated by Elements of Defect One

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Authors' contributions

This work was carried out in collaboration between all the authors. Authors ATI and MB designed the problem, formulated the mathematical statements, and wrote the first draft of the manuscript. Author ATI managed the literature searches. Author MJI came up with the research methodology. All authors read and approved the final manuscript.

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Abstract

Let $\mathcal{T}_n$ be the semigroup of full transformation on a finite set n. Then, a map $\alpha \in \mathcal{T}_n$ is said to be a contraction, if for all $x, y \in X_n$, $|x\alpha - y\alpha| \leq |x - y|$. Let $\mathcal{CT}_n$ denote the subsemigroup of all contraction maps in $\mathcal{T}_n$. In this paper we calculated the rank of the subsemigroup of $\mathcal{CT}_n$ generated by elements of defect one, where the defect of $\alpha \in \mathcal{CT}_n$ is defined to be the cardinality of the set $X_n \backslash \mathrm{im}(\alpha)$ and rank of a semigroup is the smallest number of generators for the semigroup.

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1 Introduction

Let $X_n = \{1, 2, \ldots, n\}$ be under its natural order. Let $\mathcal{S}_n$ and $\mathcal{T}_n$ denote respectively, the symmetric group consisting of all permutations of X_n, and the full transformation semigroup consisting of all transformations of X_n. The importance of the study of $\mathcal{T}_n$, as naturally occurring semigroup, is justified by its universal property that every finite semigroup is embeddable in some $\mathcal{T}_n$, which is analogous to Cayley's theorem for symmetric group $\mathcal{S}_n$. Thus, studying certain subsemigroups of $\mathcal{T}_n$ will be of great importance in the theory of semigroup. Many subsemigroups of $\mathcal{T}_n$, such as the subsemigroups of order-preserving and order-decreasing elements of $\mathcal{T}_n$, have been considered by many authors, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9].

Umar [9] showed that every element of S_n^-, the subsemigroup of all order-decreasing elements of $\mathcal{T}_n$, is expressible as a product of idempotents. [5] calculated the rank and idempotent rank of the subsemigroup of order-preserving element ($\mathcal{O}_n$) to be n and $2(n-1)$ respectively. Zhao [10] completely described maximal regular subsemibands of the two sided ideal of $\mathcal{O}_n$. See also [1, 2, 5, 11, 12, 13]. for other algebraic properties in the semigroup $\mathcal{O}_n$ and some of its subsemigroups.

A map α in $\mathcal{T}_n$ is said to be a *contraction*, if $|x\alpha - y\alpha| \le |x - y|$, for all $x, y \in X_n$. The sets of all contraction maps in $\mathcal{T}_n$, denoted by $\mathcal{CT}_n$, is a subsemigroup of $\mathcal{T}_n$. Likewise, the set of all order-preserving contraction maps in $\mathcal{T}_n$ is a subsemigroup of $\mathcal{T}_n$ and is denoted by $\mathcal{OCT}_n$. The name contraction map first appeared in [14], but algebraic and combinatorial study of the semigroup $\mathcal{OCT}_n$ were initiated in [15]. Order and regularity in the semigroup $\mathcal{CT}_n$ were investigated in [15]. Garba et al. [4] characterised Green's and starred Green's relations of the subsemigroups $\mathcal{CT}_n$ and $\mathcal{OCT}_n$. For an $\alpha \in \mathcal{T}_n$, the height and defect of α are defined to be the cardinalities of the sets $\mathrm{im}(\alpha)$ and $X_n \setminus \mathrm{im}(\alpha)$ respectively. In this paper, we obtain the rank of the subsemigroup of $\mathcal{CT}_n$ generated by elements of defect one, which is the smallest number of generators for the subsemigroup.

2 Preliminaries

Let $\mathcal{S}_n$ and $\mathcal{T}_n$ be the symmetric group and full transformation semigroup on $X_n = \{1, 2, \ldots, n\}$, respectively. Let $Sing_n = \mathcal{T}_n \setminus \mathcal{S}_n$ be the semigroup of all singular self-maps of X_n and let

$$\mathcal{CT}_n = \{\alpha \in Sing_n : (\forall x, y \in X_n)|x\alpha - y\alpha| \le |x - y|\} \tag{2.1}$$

be the subsemigroup of $Sing_n$ consisting of all contraction maps. We record the following definition and characterisation of contraction maps from [4].

Definition 2.1. Let A be a subset of X_n and let $\{A_1, A_2, \ldots, A_r\}$ be a partition of X_n. Then A will be called *convex*, if for all $x, y \in X_n$, $(x, y \in A$ and $x \le z \le y) \implies z \in A$. A is called a *transversal* of $\{A_1, A_2, \ldots, A_r\}$ if $|A| = r$ and each A_i $(1 \le i \le r)$ contains exactly one point of A. The partition $\{A_1, A_2, \ldots, A_r\}$ will be called *convex partition*, if it possesses a convex transversal.

Theorem 2.1. *Let α be an element of $\mathcal{T}_n$ of height r, where $r \le n$. Then, α is contraction if and only if*

(i) $\mathrm{im}(\alpha)$ *is a convex subset of X_n, and*

(ii) for each $i \in \mathrm{im}(\alpha)$ and each $x \in i\alpha^{-1}$, if $x - 1 \in k\alpha^{-1}$ and $x + 1 \in t\alpha^{-1}$, then $k, t \in \Phi_i$, where

$$\Phi_i = \begin{cases} \{i, i+1\} & \text{if } i = 1 \\ \{i-1, i, i+1\} & \text{if } 1 < i < r \\ \{i-1, i\} & \text{if } i = r. \end{cases}$$

3 Rank of the Subsemigroup

In this section, we determine the rank of the subsemigroup of $\mathcal{CT}_n$ generated by element of defect one. We start by noting that $\mathcal{CT}_n$ like $Sing_n$ may be partitioned into classes

$$K_1, K_2, \dots, K_{n-1},$$

where, for each $1 \leq r \leq n-1$, $K_r = \{\alpha \in \mathcal{CT}_n : |\text{im}(\alpha)| = r\}$. Our first step is to describe the subsemigroup of $\mathcal{CT}_n$ generated by elements of defect one, that is, the semigroup of $\mathcal{CT}_n$ generated by K_{n-1}. For this, we prove the following lemma.

Lemma 3.1. *For $n \geq 3$, the subsemigroup of $\mathcal{CT}_n$ generated by K_{n-1} is*

$$\langle K_{n-1} \rangle = \{\alpha \in \mathcal{CT}_n : \alpha(1) = \alpha(3) \text{ or } \alpha(n-2) = \alpha(n) \text{ or } \alpha(i) = \alpha(i+1),\ 1 \leq i \leq n-1\}.$$

Proof. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$$

be an element of height r, where $a_{i+1} = a_i + 1$, for each $1 \leq i \leq r-1$. Since $1 \leq r \leq n-2$ and $n \geq 3$, there must exist a block A_i for which $|A_i| \geq 2$. We consider three cases as follows:

Case I. If $i, i+1 \in A_j$, $1 \leq j \leq r$, then $\alpha = \beta\gamma$, where

$$\beta = \begin{pmatrix} 1 & 2 & \cdots & i-1 & \{i, i+1\} & i+2 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n-1 \end{pmatrix},$$

and for all $x \in X_n$,

$$x\gamma = \begin{cases} \alpha(x), & 1 \leq x \leq i \\ \alpha(x+1), & i+1 \leq x \leq n-1 \\ a_r + 1, & x = n. \end{cases}$$

Case II. If $1, 3 \in A_j$, $1 \leq j \leq r$, then $\alpha = \beta\gamma$, where

$$\beta = \begin{pmatrix} \{1, 3\} & 2 & 4 & \cdots & n \\ 2 & 3 & 4 & \cdots & n \end{pmatrix},$$

and for all $x \in X_n$,

$$x\gamma = \begin{cases} x, & x = 1 \\ \alpha(x-1), & x = 2, 3 \\ \alpha(x), & 4 \leq x \leq n. \end{cases}$$

Case III. If $n-2, n \in A_j$, $1 \leq j \leq r$, then $\alpha = \beta\gamma$, where

$$\beta = \begin{pmatrix} 1 & 2 & \cdots & n-3 & n-1 & \{n-2, n\} \\ 1 & 2 & \cdots & n-3 & n-2 & n-1 \end{pmatrix},$$

and for all $x \in X_n$,

$$x\gamma = \begin{cases} \alpha(x), & 1 \leq x \leq n-3 \\ \alpha(x+1), & x = n-2, n-1 \\ a_r + 1, & x = n. \end{cases}$$

Thus,

$$\langle K_{n-1} \rangle \supseteq \{\alpha \in \mathcal{CT}_n : \alpha(1) = \alpha(3) \text{ or } \alpha(n-2) = \alpha(n) \text{ or } \alpha(i) = \alpha(i+1),\ 1 \leq i \leq n-1\}.$$

In order to establish the opposite inclusion, we note that, if

$$\alpha \notin \{\alpha \in \mathcal{CT}_n : \alpha(1) = \alpha(3) \text{ or } \alpha(n-2) = \alpha(n) \text{ or } \alpha(i) = \alpha(i+1),\ 1 \leq i \leq n-1\},$$

then $\alpha \notin \langle K_{n-1} \rangle$. For if $\alpha = \beta_1\beta_2 \cdots \beta_k$ with $\beta_j \in K_{n-1}$ $(1 \leq j \leq k)$, then by Theorem 2.1, each β_j must possesses as its block $1, 3$ or $n-2, n$ or $i, i+1$ (for exactly one i with $1 \leq i \leq k$). But this clearly contradicts the choice of

$$\alpha \notin \{\alpha \in \mathcal{CT}_n : \alpha(1) = \alpha(3) \text{ or } \alpha(n-2) = \alpha(n) \text{ or } \alpha(i) = \alpha(i+1),\ 1 \leq i \leq n-1\}.$$

□

Now, since $|\text{im}(\alpha\beta)| \leq \min\{|\text{im}(\alpha), \text{im}(\beta)|\}$ (for each $\alpha \in \mathcal{CT}_n$), a set of elements in K_{n-1} generates $\langle K_{n-1} \rangle$ if and only if it generates K_{n-1}. Thus, $\text{rank}(K_{n-1}) = \text{rank}(\langle K_{n-1} \rangle)$.

Using Theorem 2.1, notice that the possible $\ker(\alpha)$-classes for elements of K_{n-1} are

$$|1,2|, |1,3|, |2,3|, |3,4|, |4,5|, \ldots, |n-2, n-1|, |n-2, n|, |n-1, n|,$$

where $|i, j|$ denotes the equivalence on X_n whose sole non-singleton class is $\{i, j\}$. Again, by Theorem 2.1, the possible image sets for elements in K_{n-1} are $X_n \setminus \{n\} = \{1, 2, \ldots, n-1\}$ and $X_n \setminus \{1\} = \{2, 3, \ldots, n\}$. Thus, since corresponding to each of the possible kernel class $|i, j|$, there are four maps in K_{n-1}, we see that $|K_{n-1}| = 4(n+1)$.

Now, let

$$\delta_{1,2} = \begin{pmatrix} \{1,2\} & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix},$$

$$\delta_{1,3} = \begin{pmatrix} 2 & \{1,3\} & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix},$$

$$\delta_{i,i+1} = \begin{pmatrix} 1 & 2 & \cdots & i-1 & \{i,i+1\} & i+2 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n-1 \end{pmatrix} \qquad (2 \leq i \leq n-2),$$

$$\delta_{n-2,n} = \begin{pmatrix} n-1 & \{n-2,n\} & n-3 & \cdots & 1 \\ 2 & 3 & 4 & \cdots & n \end{pmatrix},$$

$$\delta_{n-1,n} = \begin{pmatrix} \{n-1,n\} & n-2 & \cdots & 1 \\ 2 & 3 & \cdots & n \end{pmatrix},$$

$$\gamma_{1,2} = \begin{pmatrix} n & n-1 & \cdots & 3 & \{1,2\} \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix},$$

$$\gamma_{1,3} = \begin{pmatrix} n & n-1 & \cdots & \{1,3\} & 2 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix},$$

$$\gamma_{i,i+1} = \begin{pmatrix} 1 & 2 & \cdots & i-1 & \{i,i+1\} & i+2 & \cdots & n \\ n-1 & n-2 & \cdots & i+1 & i & i-1 & \cdots & 1 \end{pmatrix} \qquad (2 \leq i \leq n-2),$$

$$\gamma_{n-2,n} = \begin{pmatrix} n-1 & \{n-2,n\} & \cdots & 2 & 1 \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix}$$

and

$$\gamma_{n-1,n} = \begin{pmatrix} \{n-1,n\} & n-2 & \cdots & 2 & 1 \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix}.$$

Also, let

$$\delta'_{1,2} = \begin{pmatrix} n & n-1 & \cdots & 3 & \{1,2\} \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix},$$

$$\delta'_{1,3} = \begin{pmatrix} n & n-1 & \cdots & \{1,3\} & 2 \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix},$$

$$\delta'_{i,i+1} = \begin{pmatrix} n & n-1 & \cdots & i+2 & \{i,i+1\} & i-1 & \cdots & 1 \\ 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n-1 \end{pmatrix} \qquad (2 \le i \le n-2),$$

$$\delta'_{n-2,n} = \begin{pmatrix} 1 & \cdots & n-3 & \{n-2,n\} & n-1 \\ 2 & \cdots & n-2 & n-1 & n \end{pmatrix},$$

$$\delta'_{n-1,n} = \begin{pmatrix} 1 & \cdots & n-2 & \{n-1,n\} \\ 2 & \cdots & n-1 & n \end{pmatrix},$$

$$\gamma'_{1,2} = \begin{pmatrix} \{1,2\} & 3 & \cdots & n \\ 2 & 3 & \cdots & n \end{pmatrix},$$

$$\gamma'_{1,3} = \begin{pmatrix} 2 & \{1,3\} & 4\cdots & n & \\ 2 & 3 & 4 & \cdots & n \end{pmatrix},$$

$$\gamma'_{i,i+1} = \begin{pmatrix} n & n-1 & \cdots & i+2 & \{i,i+1\} & i-1 & \cdots & 1 \\ n-1 & n-2 & \cdots & i+1 & i & i-1 & \cdots & 1 \end{pmatrix} \qquad (2 \le i \le n-2),$$

$$\gamma'_{n-2,n} = \begin{pmatrix} 1 & \cdots & n-3 & \{n-2,n\} & n-1 \\ 1 & \cdots & n-3 & n-2 & n-1 \end{pmatrix}$$

and

$$\gamma'_{n-1,n} = \begin{pmatrix} 1 & \cdots & n-2 & \{n-1,n\} \\ 1 & \cdots & n-2 & n-1 \end{pmatrix}.$$

Then,

$$\begin{aligned} K_{n-1} \quad = \quad & \{\delta_{1,2}, \delta_{1,3}, \delta_{i,i+1}, \delta_{n-2,n}, \delta_{n-1,n}, \delta'_{1,2}, \delta'_{1,3}, \delta'_{i,i+1}, \delta'_{n-2,n}, \delta'_{n-1,n}, \gamma_{1,2}, \\ & \gamma_{1,3}, \gamma_{i,i+1}, \gamma_{n-2,n}, \gamma_{n-1,n}, \gamma'_{1,2}, \gamma'_{1,3}, \gamma'_{i,i+1}, \gamma'_{n-2,n}, \gamma'_{n-1,n} : 2 \le i \le n-2\}. \end{aligned}$$

Next, we have

Lemma 3.2. $K_{n-1} \subseteq \langle \delta_{1,2}, \delta_{1,3}, \delta_{2,3}, \delta_{3,4}, \ldots, \delta_{n-2,n-1}, \delta_{n-2,n}, \delta_{n-1,n} \rangle$.

Proof. This is base on a simple observation that

$$\begin{aligned} \delta_{1,2}\delta_{n-1,n} &= \gamma_{1,2}, & \delta_{1,3}\delta_{n-1,n} &= \gamma_{1,3}, \\ \delta_{n-2,n}\delta_{1,3} &= \gamma_{n-2,n}, & \delta_{n-1,n}\delta_{1,3} &= \gamma_{n-1,n}, \\ \gamma_{1,2}\delta_{1,2} &= \delta'_{1,2}, & \gamma_{1,3}\delta'_{1,2} &= \delta'_{1,3}, \\ \gamma_{n-2,n}\delta_{n-2,n} &= \delta'_{n-2,n}, & \delta_{n-1,n}\gamma_{1,2} &= \delta'_{n-1,n}, \\ \gamma^2_{1,2} &= \gamma'_{1,2}, & \gamma^2_{1,3} &= \gamma'_{1,3}, \\ \gamma^2_{n-1,n} &= \gamma'_{n-1,n}, & \gamma^2_{n-2,n} &= \gamma'_{n-2,n}, \\ \delta_{i,i+1}\delta'_{n-1,n} &= \gamma_{i,i+1}, & \gamma_{i,i+1}\delta'_{1,2} &= \delta'_{i,i+1}, \\ \delta_{i,i+1}\delta_{n-1,n} &= \gamma'_{i,i+1}. \end{aligned}$$

□

Hence we have proved the next result.

Lemma 3.3. $\text{rank}(\langle K_{n-1} \rangle) \le n+1$.

Now, it follows that any generating set for K_{n-1} (or $\langle K_{n-1} \rangle$) must pick from each of the possible Kernel classes and also, from each of the possible image sets in K_{n-1}. Thus

$$\text{rank}(\langle K_{n-1} \rangle) \ge n+1.$$

This together with Lemma 3.3 give the following theorem.

Theorem 3.4. *For any* $n \ge 3$, $\text{rank}(\langle K_{n-1} \rangle) = n+1$.

4 Conclusion

The set of all elements of height one in the semigroup $\mathcal{CT}_n$, of all full contraction maps of a finite set, has been found to generate only a proper subsemigroup of $\mathcal{CT}_n$. The subsemigroup was described and the minimum cardinality for its generating set is obtained to be equal to $n + 1$.

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Competing Interests

Authors have declared that no competing interests exist.

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