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Initial Coefficients Estimates for Certain Generalized Class of Analytic Functions Involving Sigmoid Function

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Authors' contributions

This work was carried out in collaboration between both authors. Author HJO designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author OMA managed the analyses of the study. Authors HJO and OMA managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

For function $F_{\alpha,n}(z)$ of Bazilevic type, initial coefficients $|a_{j+1}(\alpha)|$, $|a_{j+2}(\alpha)|$ and $|a_{j+3}(\alpha)|$ for certain generalized class of analytic functions involving logistic sigmoid function are obtained. Further, the Fekete-Szego functional $|a_{j+2}(\alpha)-\mu a_{j+1}^2(\alpha)|$ is also considered for functions belonging to the said class of analytic functions. Several other results follow as simple consequences.

Keywords: Analytic; starlike; convex; Bazilevic; sigmoid function.

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1 Introduction

The theory of analytic functions has wide application in many physical problems such as in heat conduction, electrostatic potential and so on. So also is the theory of special function such as sigmoid functions.

Conventionally, it is believed that activation function is an information process that is inspired by the way nervous system like brain, process information. In fact, it is composed of large number of highly interconnected processing element (neurons) working to solve a definite task. This function works in similar way the brain does. The most widely used sigmoid function is the logistic function which has a lower bound of zero (0) and upper bound of one (1).

Many sigmoid functions have power series expansion which alternate in sign while some have inverse with hypergeometric series expansion. They can be evaluated differently especially by truncated series expansion. The logistic sigmoid function is defined as

$$g(z) = \frac{1}{1+e^{-z}} = \frac{1}{2} + \frac{1}{4}z - \frac{1}{48}z^3 + \frac{1}{480}z^5 \tag{1.1}$$

and has the following properties:

(i) It outputs real number between 0 and 1
(ii) It maps a very large input domain to a small range of outputs
(iii) It never loses information because it is a one-to-one function
(iv) It increases monotonically.

In view of the above properties sigmoid function is highly referred in geometric Function Theory (See [1]).

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \qquad z \in D \tag{1.2}$$

which are analytic in the open unit disk $D = \{z; |z| < 1\}$ and normalized by

$f(z) = f^1(0) - 1 = 0$. Also, let *S* denote the subclass of *A* that are normalized and univalent in *D*. Now, for function $f(z)$ of the form (1.2), one can write that

$$f(z)^{\alpha} = z^{\alpha} + \sum_{k=j+1}^{\infty} a_k(\alpha) z^{\alpha+k-1} \qquad \alpha > 0, \ z \in D \tag{1.3}$$

Using Salagean derivative operator [2], one can equally write that

$$D^n f(z)^{\alpha} = \alpha^n z^{\alpha} + \sum_{k=j+1}^{\infty} (\alpha+k-1)^n a_k(\alpha) z^{\alpha+k-1} \qquad \alpha > 0, \ z \in D \tag{1.4}$$

In 1992, Abdulhalim [3] introduced a generalization of certain family of Bazilevic functions satisfying inequality

$$\Re e\left\{\frac{D^n f(z)^\alpha}{z^\alpha}\right\}>0, \qquad \alpha>0, \; z\in D. \tag{1.5}$$

where the parameter α and the operator D^n is the famous Sălăgean derivative operator [2] defined by $D^n f(z)=D(D^{n-1}f(z))=z(D^{n-1}f(z))'$ are defined as earlier. He denoted this class of functions by $B_n(\alpha)$. It is easily seen that his generalization has extraneously included analytic functions satisfying

$$\Re e\left\{\frac{f(z)^\alpha}{z^\alpha}\right\}, \quad z\in D \tag{1.6}$$

which largely non-univalent in the unit disk. By proving the inclusion

$$B_{n+1}(\alpha)\subset B_n(\alpha). \tag{1.7}$$

In 1994, Opoola [4] further studied a more generalized form of (1.5) and denoted it by $T_n^\alpha(\gamma)$ (Bazilevic class of order gamma) such that

$$\left\{\frac{D^n f(z)^\alpha}{z^\alpha}\right\}>\gamma, \qquad \alpha>0, \; z\in D. \tag{1.8}$$

Recently, a little modification was made to (1.8) such that

$$\Re e\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\}>\beta \qquad \alpha>0, \; z\in D\,.$$

For the purpose of the present investigation, let $F_{\alpha,n}(z)\in T_n^\alpha$ be defined as follows

$$F_{\alpha,n}(z)=z\left(1+\sum_{k=j+1}^{\infty}\alpha_{n,k}a_k(\alpha)z^{k-1}\right), \; j\in N \tag{1.9}$$

where

$$\alpha_{n,k}=\left(\frac{\alpha+k-1}{\alpha}\right)^n. \tag{1.10}$$

Using (1.9), we give the following definition.

Definition:

Let $F_{\alpha,n}(z)\in S_{\delta,\gamma}^{\alpha,n}(\beta,\theta,\lambda,j)$ then,

$$\Re e\left\{\left(1+\delta^2\right)\left(\frac{e^{i\theta}\, z\left(F'_{\alpha,n}(z)\right)^{\lambda}+\left(2\delta^2-\delta\right)z^2\left[\left(F'_{\alpha,n}(z)\right)^{\lambda}\right]'}{4(\delta-\delta^2)z+\left(2\delta^2-\delta\right)zF'_{\alpha,n}(z)+\left(2\delta^2-3\delta+1\right)F_{\alpha,n}(z)}-\gamma\right)\right\}$$
$$>\beta\left|\left(1+\delta^2\right)\left(\frac{e^{i\theta}\, z\left(F'_{\alpha,n}(z)\right)^{\lambda}+\left(2\delta^2-\delta\right)z\left[\left(F'_{\alpha,n}(z)\right)^{\lambda}\right]'}{4(\delta-\delta^2)z+\left(2\delta^2-\delta\right)zF'_{\alpha,n}(z)+\left(2\delta^2-3\delta+1\right)F_{\alpha,n}(z)}-1\right)\right| \tag{1.11}$$

for $\alpha>0,\ \beta\geq 0,\ 0\leq\delta\leq 1,\ \lambda\geq 1, -1\leq\gamma<1,\ n\in N_0=N\cup\{0\}, n\geq j$ and $0\leq\theta<\frac{\pi}{2}$.

It is observed that $TS^{\alpha,n}_{\delta,\gamma}(\lambda,\theta,\alpha,j)=S^{\alpha,n}_{\delta,\gamma}(\lambda,\theta,\alpha,j)\cap T$, where T is the subclass of S consisting of functions of the form

$$F_{\alpha,n}(z)=z-\sum_{k=j+1}^{\infty}a_k z^k, \qquad a_k>0\ \forall\ j+1. \tag{1.12}$$

With various choices of parameters involved, several subclasses of analytic functions (known and new) are obtained. Few of them are given below:

1. Let $F_{\alpha,n}(z)\in S^{1,0}_{0,\gamma}(\beta,\theta,\lambda,j)$. Then,

$$\Re e\left\{\left(\frac{e^{i\theta}\, z\left(F'_{1,0}(z)\right)^{\lambda}}{F_{1,0}(z)}-\gamma\right)\right\}>\beta\left|\left(\frac{e^{i\theta}\, z\left(F'_{1,0}(z)\right)^{\lambda}}{F_{1,0}(z)}-1\right)\right|$$

which is the λ *–pseudo-spiralike class of order* γ.

2. Let $F_{\alpha,n}(z)\in S^{1,0}_{0,\gamma}(\beta,0,\lambda,j)$. Then,

$$\Re e\left\{\left(\frac{z\left(F'_{1,0}(z)\right)^{\lambda}}{F_{1,0}(z)}-\gamma\right)\right\}>\beta\left|\left(\frac{z\left(F'_{1,0}(z)\right)^{\lambda}}{F_{1,0}(z)}-1\right)\right|$$

which is the λ *–pseudo-starlike class of order* γ.

3. Let $F_{\alpha,n}(z)\in S^{1,0}_{0,\gamma}(\beta,0,1,j)$. Then,

$$\Re e\left\{\left(\frac{z\left(F'_{1,0}(z)\right)}{F_{1,0}(z)}-\gamma\right)\right\}>\beta\left|\left(\frac{z\left(F'_{1,0}(z)\right)}{F_{1,0}(z)}-1\right)\right|$$

which is the *starlike class of order* γ.

4. Let $F_{\alpha,n}(z) \in S_{1,\gamma}^{1,0}(0,0,1,j)$. Then,

$$\Re e\left\{\left(1+\frac{z\left[\left(F_{1,0}'(z)\right)^{\lambda}\right]'}{F_{1,0}'(z)}-\gamma\right)\right\}>0$$

which is the *convex class of order* γ.

5. Let $F_{\alpha,n}(z) \in S_{0,\gamma}^{1,0}(\beta,\theta,2,j)$. Then,

$$\Re e\left\{\left(F_{1,0}'(z)\cdot\frac{e^{i\theta}\, z\left(F_{1,0}'(z)\right)}{F_{1,0}(z)}-\gamma\right)\right\}>\beta\left|\left(F_{1,0}'(z)\cdot\frac{e^{i\theta}\, z\left(F_{1,0}'(z)\right)}{F_{1,0}(z)}-1\right)\right|$$

which is the product of a combination of geometric expression for *bounded turning function and spiralike class of order* γ.

6. Let $F_{\alpha,n}(z) \in S_{0,\gamma}^{1,0}(\beta,0,2,j)$. Then,

$$\Re e\left\{\left(F_{\alpha,n}'(z).\frac{z\left(F_{1,0}'(z)\right)}{F_{1,0}(z)}-\gamma\right)\right\}>\beta\left|\left(F_{\alpha,n}'(z).\,\frac{z\left(F_{1,0}'(z)\right)}{F_{1,0}(z)}-1\right)\right|$$

which is the product of a combination of geometric expression for *bounded turning function and starlike class of order* γ.

Remark A:

By careful selection of the values for the parameters involves several interesting subclasses known and new are obtained (see [5,6,7,8,9,10,11,12,13]).

The primary aim of the present paper is to investigate the relationship between sigmoid function and certain generalized class of analytic functions in terms of coefficients bounds. The paper further examines the Fekete-Szego problem involving this generalized class and sigmoid function.

2 Main Results

Lemma (2.1) Let g be sigmoid function of the form (1.1). Then, let $\phi(z) = 2g(z)$ such that

$$\phi(z)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}z^m\right)^k. \tag{2.1}$$

Then $\phi(z) \in P$, $|z|<1$ where P is the class of Caratheodory functions and $\phi(z)$ denotes the celebrated sigmoid function (see [14]).

Lemma (2.2) Let

$$\phi(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m \right)^k \tag{2.2}$$

Then, $|\phi(z)| < 2$ (see [14]).

Lemma (2.3) Let $\phi(z) \in P$ and is starlike, then it is a normalized univalent function of the form *(1.2)* (see [1]).

Remark B:

Suppose that k = 1 and also let $\phi(z) = 1 + \sum_{m=1}^{\infty} C_m z^m$

where $C_m = \frac{(-1)^{m+1}}{2m!}$, then $|C_m| \leq 2, \quad m = 1, 2, \ldots$.

This result is the best possible (see [14]).

Theorem (2.4): Let $F_{\alpha,n}(z) \in S_{\delta,\gamma}^{\alpha,n}(\phi, \beta, \theta, \lambda, j)$. Then,

$$|a_{j+1}(\alpha)| \leq \frac{(1+\delta^2)\psi}{2M_1 \alpha_{n,\, j+1}} \tag{2.3}$$

for $\alpha > 0, \lambda \geq 1, \beta \geq 0, 0 \leq \delta < 1, n \in N_0, n \geq j, 0 \leq \theta < \frac{\pi}{2}$, where $\alpha_{n,\, j+1} = \left(\frac{\alpha + j}{\alpha} \right)^n$,

$M_1 = \lambda(j+1)\left(e^{i\theta} + j(2\delta^2 - \delta)\right) - e^{io}(1-\beta)\left((j+1)(2\delta^2 - \delta) + 2\delta^2 - 3\delta + 1\right)$ and

$\psi = \left(e^{i\theta}(1-\beta) + (\beta - \gamma)(1+\delta^2)\right)$.

$$|a_{j+2}(\alpha)| \leq \frac{(1+\delta^2)\psi^2 \begin{Bmatrix} 2\left((j+1)(2\delta^2 - \delta) + 2\delta^2 - 3\delta + 1\right)M_1 \\ -(1+\delta^2)^2(1-\beta)\left(e^{i\theta} + (j+1)(2\delta^2 - \delta)\right)\lambda(\lambda - 1)(j+1)^2 \end{Bmatrix}}{8M_1^2 M_2 \alpha_{n,\, j+2}} \tag{2.4}$$

where

$M_2 = \lambda(j+2)(1+\delta^2)(1-\beta)\left(e^{i\theta} + (j+1)(2\delta^2 - \delta)\right) - e^{i\theta}(1-\beta)\left((j+2)(2\delta^2 - \delta) + 2\delta^2 - 3\delta + 1\right)$,

$\alpha_{n,\, j+2} = \left(\frac{\alpha + j + 1}{\alpha} \right)^n$, ψ and M_1, are as earlier defined.

$$\left|a_{j+3}(\alpha)\right| \le \frac{\left(1+\delta^2\right)\psi}{24M_3\,\alpha_{n,j+3}} + \frac{(1+\delta^2)\psi^3}{48M_1^3M_2M_3\,\alpha_{n,j+3}}\left\{\begin{array}{l}\left(\begin{array}{l}3M_1\left[(j+2)(2\delta^2-\delta)+2\delta^2-3\delta+1\right]\\-3(\lambda^2\lambda)(j+1)(j+2)\end{array}\right)\\ \left(\begin{array}{l}2M_1\left[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1\right]\\-(1+\delta^2)^2(1+j)^2(1-\beta)(\lambda^2-\lambda)\\ \left[e^{i\theta}+(j+1)(2\delta^2-\delta)\right]\end{array}\right)\\ -M_2\left(1+\delta^2\right)(j+1)^3\left(\lambda^3-3\lambda^2+2\lambda\right)\end{array}\right\} \tag{2.5}$$

where

$M_3=\lambda(j+3)\left(1+\delta^2\right)(1-\beta)\left[e^{i\theta}+(j+2)\left(2\delta^2-\delta\right)\right]-e^{i\theta}(1-\beta)\left[(j+3)\left(2\delta^2-\delta\right)+2\delta^2-3\delta+1\right]$,

$\alpha_{n,j+3}=\left(\dfrac{\alpha+j+2}{\alpha}\right)^n$ and ψ, M_1, M_2 are as earlier defined.

Proof: Let $F_{\alpha,n}(z)\in S_{\delta,\gamma}^{\alpha,n}(\beta,\theta,\lambda,j)$. Then, by definition there exists $\phi(z)\in P$ such that

$$\frac{\dfrac{\left(1+\delta^2\right)(1-\beta)\left\{e^{i\theta}z\left(F'_{\alpha,n}(z)\right)^\lambda+\left(2\delta^2-\delta\right)z^2\left[\left(F'_{\alpha,n}(z)\right)^\lambda\right]'\right\}}{4\left(\delta-\delta^2\right)z+\left(2\delta^2-\delta\right)zF'_{\alpha,n}(z)+\left(2\delta^2-3\delta+1\right)F'_{\alpha,n}(z)}-\left(1+\delta^2\right)(\gamma-\beta)}{\left[e^{i\theta}(1-\beta)+(\beta-\delta)\left(1+\delta^2\right)\right]}=\phi(z) \tag{2.6}$$

$$\left(1+\delta^2\right)(1-\beta)\left\{e^{i\theta}z\left(F'_{\alpha,n}(z)\right)^\lambda+\left(2\delta^2-\delta\right)z^2\left[\left(F'_{\alpha,n}(z)\right)^\lambda\right]'\right\}$$

$$=\left(\begin{array}{l}4\left(\delta-\delta^2\right)z+\left(2\delta^2-\delta\right)zF'_{\alpha,n}(z)\\+\left(2\delta^2-3\delta+1\right)F'_{\alpha,n}(z)\end{array}\right)\left(\left(1+\delta^2\right)(\gamma-\beta)+\phi(z)\left[\begin{array}{l}e^{i\theta}(1-\beta)\\+(\beta-\delta)\left(1+\delta^2\right)\end{array}\right]\right) \tag{2.7}$$

where the function $\phi(z)$ is the modified sigmoid function given by

$$\phi(z)=1+\tfrac{1}{2}z^j-\tfrac{1}{24}z^{j+2}+\tfrac{1}{240}z^{j+4}-\tfrac{1}{64}z^{j+5}+\frac{779}{20160}z^{j+6}-\ldots \tag{2.8}$$

Equating the coefficients of the like powers of z^j, z^{j+1}, z^{j+2} and z^{j+3} in the series expansion of (2.7), then

$$\left\{\begin{array}{l}\left(1+\delta^2\right)(1-\beta)[e^{i\theta}+j\lambda(2\delta^2-\delta)(j+1)]\\-e^{i\theta}(1-\beta)\left[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1\right]\end{array}\right\}\alpha_{n,j+1}a_{j+1}(\alpha)=\frac{1}{2}\left(1+\delta^2\right)\left[\begin{array}{l}e^{i\theta}(1-\beta)\\+(\beta-\gamma)\left(1+\delta^2\right)\end{array}\right] \tag{2.9}$$

$$\left\{\begin{array}{l}\lambda(1+\delta^2)(1-\beta)(j+2)\left[e^{i\theta}+(j+1)(2\delta^2-\delta)\right] \\ -e^{i\theta}(1-\beta)\begin{bmatrix}(j+2)(2\delta^2-\delta) \\ +2\partial^2-3\partial+1\end{bmatrix}\end{array}\right\}\alpha_{n,j+2}a_{n,j+2}(\alpha)$$

$$=\frac{1}{2}\left[e^{i\theta}(1-\beta)+(\beta-\delta)(1+\delta^2)\right]\begin{bmatrix}(j+1)(2\delta^2-\delta) \\ +2\delta^2-3\delta+1\end{bmatrix}\alpha_{n,j+1}a_{n,j+1}(\alpha) \qquad (2.10)$$

$$-(1+\delta^2)(1-\beta)(j+1)^2\frac{\lambda(\lambda-1)}{2}\left[e^{i\theta}+(j+1)(2\delta^2-\delta)\right]\alpha^2{}_{n,j+1}a^2_{j+1}(\alpha)$$

$$\left\{\begin{array}{l}\lambda(j+3)(1+\delta^2)(1-\beta)\left[e^{i\theta}+(j+2)(2\delta^2-\delta)\right] \\ -e^{i\theta}(1-\beta)\left[(j+3)\begin{pmatrix}2\delta^2-\delta) \\ +2\delta^2-3\delta+1\end{pmatrix}\right]\end{array}\right\}\alpha_{n,j+3}a_{n,j+3}(\alpha)$$

$$=\frac{1}{2}\left[e^{i\theta}(1-\beta)+(\beta-\gamma)(1+\delta^2)\right]\left[(j+2)(2\delta^2-\delta)+2\delta^2-3\delta+1\right]\alpha_{n,j+2}a_{j+2}(\alpha) \qquad (2.11)$$

$$-\frac{1}{24}(1+\delta^2)\left[e^{i\theta}(1-\beta)+(\beta-\gamma)(j+\delta^2)\right]-\frac{\lambda(\lambda-1)(\lambda-2)}{6}(j+1)^3\alpha^3_{n,j+1}a^3_{j+1}(\alpha)$$

$$-\lambda(\lambda-1)(j+1)(j+2)\alpha_{n,j+1}\alpha_{n,j+2}a_{j+1}(\alpha)a_{j+2}(\alpha)$$

Thus, the inequalities (2.3), (2.4) and (2.5) follow respectively, from (2.9), (2.10) and (2.11) and this completes the proof of theorem (2.4).

Corollary 2.5: Let $F_{\alpha,n}(z)\in S^{1,0}_{0,\gamma}(\phi,0,\theta,\lambda,j)$. Then,

$$|a_{j+1}|\le\frac{e^{i\theta}-\gamma}{2e^{i\theta}[(\lambda(j+1)-1)]},\qquad |a_{j+2}|\le\frac{(e^{i\theta}-\gamma)^2\left(2[\lambda(j+1)-1]-(\lambda^2-\lambda)(j+1)^2\right)}{8\left[e^{i\theta}(\lambda(j+1)-1)\right]^2[(\lambda(j+2)-1)]},$$

and

$$|a_{j+3}|\le\frac{(e^{i\theta}-\gamma)}{24\left[e^{i\theta}(\lambda(j+3)-1)\right]}+\frac{(e^{i\theta}-\gamma)^3\left\{\begin{array}{l}\left(3e^{i\theta}[\lambda(j+1)-1]-(j+1)(j+2)(\lambda^2-\lambda)\right) \\ \left(2[\lambda(j+1)-1]-(j+1)^2(\lambda^2-\lambda)\right) \\ -(j+1)^3[\lambda(j+2)-1][\lambda^3-3\lambda^2+2\lambda]\end{array}\right\}}{48\left[e^{i\theta}(\lambda(j+1)-1)\right]^3\left[e^{i\theta}(\lambda(j+2)-1)\right][(\lambda(j+3)-1)]}.$$

Corollary 2.6: Let $F_{\alpha,n}(z)\in S^{1,0}_{0,\gamma}(\phi,0,0,\lambda,j)$. Then,

$$|a_{j+1}|\le\frac{1-\gamma}{2[(\lambda(j+1)-1)]},\qquad |a_{j+2}|\le\frac{(1-\gamma)^2\left(2[\lambda(j+1)-1]-(\lambda^2-\lambda)(j+1)^2\right)}{8[(\lambda(j+1)-1)]^2[(\lambda(j+2)-1)]},$$

and

$$|a_{j+3}| \leq \frac{(1-\gamma)}{24[(\lambda(j+3)-1)]} + \frac{(1-\gamma)^3 \left\{ \begin{matrix} (3\,[\lambda(j+1)-1]-(j+1)(j+2)(\lambda^2-\lambda)) \\ (2[\lambda(j+1)-1]-(j+1)^2(\lambda^2-\lambda)) \\ -(j+1)^3[\lambda(j+2)-1][\lambda^3-3\lambda^2+2\lambda] \end{matrix} \right\}}{48[(\lambda(j+1)-1)]^3\,[(\lambda(j+2)-1)][(\lambda(j+3)-1)]}.$$

Corollary 2.7: Let $F_{\alpha,n}(z) \in S_{0,\gamma}^{1,0}(\phi,0,0,\lambda,1)$. Then,

$$|a_2| \leq \frac{1-\gamma}{2(2\lambda-1)}, \qquad |a_3| \leq \frac{(1-\gamma)^2\,(4\lambda-2\lambda^2-1)}{4(2\lambda-1)^2\,(3\lambda-1)},$$

and

$$|a_4| \leq \frac{(1-\gamma)}{24(4\lambda-1)} + \frac{(1-\gamma)^3\,(24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3)}{24(2\lambda-1)^3\,(3\lambda-1)(4\lambda-1)}.$$

Incidentally, this result in (2.7) coincides with that of Murugusundaramoorthy and Janani [9].

Corollary 2.8: Let $F_{\alpha,n}(z) \in S_{0,\gamma}^{1,0}(\phi,0,0,1,1)$. Then,

$$|a_2| \leq \frac{1-\gamma}{2}, \quad |a_3| \leq \frac{(1-\gamma)^2}{8}, \quad |a_4| \leq \frac{1-\gamma}{72} + \frac{(1-\gamma)^3}{48}$$

Corollary 2.9: Let $F_{\alpha,n}(z) \in S_{0,0}^{1,0}(\phi,0,0,1,1)$. Then,

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{8}, \quad |a_4| \leq \frac{5}{144}.$$

Corollary 2.10: Let $F_{\alpha,n}(z) \in S_{0,0}^{1,0}(\phi,0,0,2,1)$. Then,

$$|a_2| \leq \frac{1}{6}, \quad |a_3| \leq \frac{1}{180}, \quad |a_4| \leq \frac{1}{840}.$$

Theorem (2.11): Let $F_{\alpha,n}(z) \in S_{\delta,\gamma}^{\alpha,n}(\phi,\beta,\theta,\lambda,j)$. Then,

$$|a_{j+2}(\alpha)-\mu a_{j+1}^2(\alpha)| \leq \frac{(1+\delta^2)\psi}{8M_1^2 M_2\,\alpha_{n,j+1}^2\,\alpha_{n,j+2}} \left| \begin{matrix} \left(\begin{matrix} 2((j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1)M_1 \\ -\lambda(\lambda-1)(j+1)^2(1+\delta^2)^2(1-\beta) \\ (e^{i\theta}+(j+1)(2\delta^2-\delta)) \end{matrix} \right) \alpha_{n,j+1}^2 \\ -2\mu M_2\,\alpha_{n,j+2} \end{matrix} \right| \quad (2.12)$$

for $\alpha > 0, \lambda \geq 1, \beta \geq 0, 0 \leq \delta < 1, n \in N_0, n \geq j, 0 \leq \theta < \frac{\pi}{2}$, where $\alpha_{n,j+1}, \psi, M_1$ and M_2, are as earlier defined.

Proof: Using (2.9) and (2.10), the result is immediate.

Corollary (2.12): Let $F_{\alpha,n}(z) \in S_{0,\gamma}^{1,0}(\phi, 0, \theta, \lambda, j)$. Then,

$$\left|a_{j+2}(\alpha) - \mu a_{j+1}^2(\alpha)\right| \leq \frac{(e^{i\theta} - \gamma)^2 \left|\left(2[\lambda(j+1)-1] - (j+1)^2(\lambda^2 - \lambda)\right) - 2\mu[\lambda(j+2)-1]\right|}{8e^{i2\theta}[\lambda(j+1)-1]^2[\lambda(j+2)-1]}$$

Corollary (2.12): Let $F_{\alpha,n}(z) \in S_{0,\gamma}^{1,0}(\phi, 0, 0, \lambda, 1)$. Then,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(1-\gamma)^2}{4(2\lambda - 1)^2}\left|\frac{4\lambda - 2\lambda^2 - 1}{3\lambda - 1} - \mu\right|$$

Incidentally, this result in (2.12) coincides with that of Murugusundaramoorthy and Janani [9].

Corollary (2.13): Let $F_{\alpha,n}(z) \in S_{0,0}^{1,0}(\phi, 0, 0, 1, 1)$. Then,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{8}\left|1 - 2\mu\right| .$$

If $\mu = 1$ in corollary 2.13 above, then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{8}.$$

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Authors have declared that no competing interests exist.

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