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On Geodesic Paracontact CR-lightlike Submanifolds

Bilal Eftal Acet[1*] and Selcen Yüksel Perktaş[1]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Adıyaman University, Adıyaman, Turkey.*

Authors' contributions

This work was carried out in collaboration between both authors. Author BEA provided the concept and designed the study then wrote the first draft of the manuscript. Author SYP managed the analyses of the study and literature searches. Both authors read and approved the final manuscript.

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Abstract

In present paper we investigate geodesic paracontact CR-lightlike submanifolds of para-Sasakian manifolds. Also some geometric results on paracontact screen CR-lightlike submanifolds are given.

Keywords: Para-sasakian manifolds; CR-lightlike submanifolds.

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1 Introduction

Because of the fact that the normal vector bundle has non-trivial intersection with the tangent vector bundle, the differential geometry of lightlike submanifolds is different from non-degenerate submanifolds. So, one can not use the classical submanifold theory for lightlike (null) submanifolds. To overcome this problem K. L. Duggal and A. Bejancu were introduced some new methods and studied lightlike submanifolds [1] (see also [2]).

*Corresponding author: E-mail: eacet@adiyaman.edu.tr;

In [1], CR-lightlike submanifolds were studied by K. L. Duggal and A. Bejancu in indefinite Kähler manifolds, then B. Şahin, R. Güneş [3] investigated geodesic CR-lightlike submanifolds of indefinite Kähler manifolds. Geodesic lightlike submanifolds of indefinite Sasakian manifolds were introduced by J. Dong, X. Liu [4]. Also, in indefinite Sasakian manifolds screen CR-lightlike submanifolds and contact CR-lightlike submanifolds were presented by K. L. Duggal, B. Şahin [5]. Since then an extensive literature has been created on lightlike submanifolds (see [6, 7, 8, 9, 10]).

On a semi-Riemannian manifold M^{2n+1}, S. Kaneyuki, M. Konzai [11] introduced a structure which is called the almost paracontact structure and then they characterized the almost paracomplex structure on $M^{2n+1} \times \mathbb{R}$. Recently, S. Zamkovoy [12] studied paracontact metric manifolds and some subclasses which is known para-Sasakian manifolds. The study of paracontact geometry has been continued by several papers ([13, 14, 15, 16, 17]) which are contained role of paracontact geometry about semi-Riemannian geometry, mathematical physics and relationships with the para-Kähler manifolds.

In this manuscript we study the lightlike submanifolds of para-Sasakian manifolds and obtain several geometric results. The paper arranged as follow. In Section 2, we recall some basic facts about lightlike submanifolds and almost paracontact metric manifolds, respectively. In Section 3, we introduce geodesic paracontact CR-lightlike submanifolds of para-Sasakian manifolds and obtain some necessary and sufficient conditions for totally geodesic, $\tilde{D}-$geodesic, $\breve{D}-$geodesic and mixed geodesic paracontact CR-lightlike submanifolds. Section 4 is devoted to paracontact screen CR-lightlike submanifolds of para-Sasakian manifolds.

2 Preliminaries

2.1 Lightlike submanifolds

Let $(\bar{M}^{n+m}, \bar{g})$ be a semi-Riemannian manifold with index q, such that $m, n \geqslant 1, 1 \leq q \leq m+n-1$ and (M^m, g) be a submanifold of $\bar{M}$, where g induced metric from $\bar{g}$ on M. In this case, M is called a *lightlike* (*null*) *submanifold* of $\bar{M}$ if g is degenerate on M. Now let us consider a degenerate metric g on M. Thus $TM^{\perp}$ is a degenerate $n-$dimensional subspace of $T_x\bar{M}$ and orthogonal subspaces T_xM and $T_xM^{\perp}$ are degenerate but no longer complementary. So, there exists a subspace $Rad\,T_xM = T_xM \cap T_xM^{\perp}$ which is called radical space. If the mapping $Rad\,TM : x \in M \to RadT_xM$, defines a smooth distribution, named *Radical distribution*, on M of rank $r > 0$ then the submanifold M is called an $r-$*lightlike submanifold* [1].

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad\,TM$ in TM. So one can write

$$TM = S(TM) \perp Rad\,TM \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad\,TM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and $Rad\,TM$ in $S(TM^{\perp})^{\perp}$, respectively. In this case we arrive at

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = \{Rad\,TM \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \tag{2.3}$$

Theorem 2.1. *[1] Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exist a complementary vector subbundle $ltr(TM)$ of $Rad\,TM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_U$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_U$, where U is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, E_i) = 1, \qquad \bar{g}(N_i, N_j) = 0, \tag{2.4}$$

where $\Gamma(Rad\,TM)$=*Span*$\{E_1, E_2, ..., E_n\}$.

For a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ we have following four cases:

* If $r < min\{m, n\}$ then M is a $r-$*lightlike* submanifold,
* If $r = n < m$, $S(TM^{\perp}) = \{0\}$ then M is a *coisotropic lightlike* submanifold,
* If $r = m < n$, $S(TM) = \{0\}$ then M is a *isotropic lightlike* submanifold,
* If $r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$ then M is a *totally null* submanifold.

By use of (2.3), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^t_X U, \quad \forall X \in \Gamma(TM),\ U \in \Gamma(tr(TM)), \tag{2.6}$$

where $\{\nabla_X Y, A_U X\}$ belongs to $\Gamma(TM)$ and $\{h(X, Y), \nabla^t_X U\}$ belongs to $\Gamma(tr(TM))$. $\bar{\nabla}$ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively.

In view of (2.2), we consider the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$. Therefore (2.5) and (2.6) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad X, Y \in \Gamma(TM), \tag{2.7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad X \in \Gamma(TM), N \in \Gamma(ltr(TM)), \tag{2.8}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \quad X \in \Gamma(TM), W \in \Gamma(S(TM^{\perp})), \tag{2.9}$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $\nabla^l_X N$, $D^l(X, W) \in \Gamma(ltr(TM))$, $\nabla^s_X W$, $D^s(X, N) \in \Gamma(S(TM^{\perp}))$ and $\nabla_X Y$, $A_N X$, $A_W X \in \Gamma(TM)$.

Assume that P is a projection of TM on $S(TM)$, then by use of (2.1), we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \qquad X, Y \in \Gamma(TM), \tag{2.10}$$

$$\nabla_X E = -A^*_E X + \nabla^{*t}_X E, \quad X \in \Gamma(TM),\ E \in \Gamma(RadTM), \tag{2.11}$$

where $\{\nabla^*_X PY, A^*_E X\}$ belong to $\Gamma(S(TM))$ and $\{h^*(X, PY), \nabla^{*t}_X E\}$ belong to $\Gamma(RadTM)$.

Using (2.10) and (2.11), we get

$$\bar{g}(h^*(X, PY), N) = \bar{g}(A_N X, PY), \tag{2.12}$$

$$\bar{g}(h^l(X, PY), E) = \bar{g}(A^*_E X, PY), \tag{2.13}$$

$$\bar{g}(h^l(X, E), E) = 0, \quad A^*_E E = 0. \tag{2.14}$$

In genereal, the induced connection ∇ on M is not metric connection. By using property of $\bar{\nabla}$ and (2.7), we have

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y). \tag{2.15}$$

However, ∇^* is a metric connection on $S(TM)$.

2.2 Almost paracontact metric manifolds

A paracontact manifold $\bar{M}^{2n+1}$ is smooth manifold equipped with a $1-$form η, a characteristic vector field ξ and a tensor field $\bar{\phi}$ of type $(1, 1)$ such that [11]:

$$\eta(\xi) = 1, \tag{2.16}$$

$$\bar{\phi}^2 = I - \eta \otimes \xi, \tag{2.17}$$

$$\bar{\phi}\xi = 0, \tag{2.18}$$

$$\eta \circ \bar{\phi} = 0. \tag{2.19}$$

If we set $D = \ker \eta = \{X \in \Gamma(T\bar{M}) : \ \eta(X) = 0\}$, then $\bar{\phi}$ induces an almost paracomplex structure on the codimension 1 distribution defined by D [11].

Moreover if the manifold $\bar{M}$ is equipped with a semi-Riemannian metric $\bar{g}$ of signature $(n+1, n)$ which is called *compatible metric* satisfying [12]

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = -\bar{g}(X, Y) + \eta(X)\eta(Y), \quad X, Y \in \Gamma(T\bar{M}) \tag{2.20}$$

then we say that $\bar{M}$ is an *almost paracontact metric manifold* with an *almost paracontact metric structure* $(\bar{\phi}, \xi, \eta, \bar{g})$.

From the definition, one can see that [12]

$$\bar{g}(\bar{\phi}X, Y) = -\bar{g}(X, \bar{\phi}Y), \tag{2.21}$$

$$\bar{g}(X, \xi) = \eta(X). \tag{2.22}$$

If $\bar{g}(X, \bar{\phi}Y) = d\eta(X, Y)$ then the almost paracontact metric manifold is said to be a *paracontact metric manifold.*

For an almost paracontact metric manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$, one can always find a local orthonormal basis $(X_i, \bar{\phi}X_i, \xi)$, $i = 1, 2, ..., n$, which is called $\bar{\phi}-$*basis* [12].

An almost paracontact metric structure $(\bar{\phi}, \xi, \eta, \bar{g})$ is a *para-Sasakian manifold* if and only if [12]

$$(\bar{\nabla}_X \bar{\phi})Y = -\bar{g}(X, Y)\xi + \eta(Y)X, \quad X, Y \in \Gamma(T\bar{M}), \tag{2.23}$$

where $\bar{\nabla}$ is a Levi-Civita connection on $\bar{M}$.

From (2.23), one arrive at

$$\bar{\nabla}_X \xi = -\bar{\phi}X. \tag{2.24}$$

Example 2.2. *[18] Let $\bar{M} = \mathbb{R}^{2n+1}$ be the $(2n+1)-$dimensional real number space with standard coordinate system $(x_1, y_1, x_2, y_2, ..., x_n, y_n, z)$. Defining*

$$\begin{aligned} \overline{\phi}\frac{\partial}{\partial x_\alpha} &= \frac{\partial}{\partial y_\alpha}, \qquad \overline{\phi}\frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial x_\alpha}, \qquad \overline{\phi}\frac{\partial}{\partial z} = 0, \\ \xi &= \frac{\partial}{\partial z}, \qquad \overline{\eta} = dz, \\ \overline{g} &= \eta \otimes \eta + \sum_{\alpha=1}^{n} (dx_\alpha \otimes dx_\alpha - dy_\alpha \otimes dy_\alpha), \end{aligned} \tag{2.25}$$

where $\alpha = 1, 2, ..., n$, then the set $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is an almost paracontact metric manifold.

3 Geodesic Paracontact CR-lightlike Submanifolds

Definition 3.1. [19] Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of a para-Sasakian manifold $(\bar{M}, \bar{g})$ such that $\xi \in \Gamma(TM)$. If the following conditions are satisfied on M, then M is called a paracontact CR-lightlike submanifold,

i) $RadTM$ is a distribution on M such that

$$RadTM \cap \bar{\phi}RadTM = \{0\},$$

ii) There exist vector bundles D_0 and $D^{'}$ on M such that

$$S(TM) = \{\bar{\phi}RadTM \oplus D^{'}\}\perp D_0 \perp \{\xi\}, \tag{3.1}$$

$$\bar{\phi}D_0 = D_0, \qquad \bar{\phi}D^{'} = L_1 \perp L_2, \tag{3.2}$$

where D_0 is non-degenerate and $L_1 = ltr(TM)$, L_2 is a vector subbundle of $S(TM^{\perp})$. So one has the following:

$$\begin{aligned} TM &= \{D \oplus D^{'}\}\perp\{\xi\}, \\ D &= RadTM \perp \bar{\phi}RadTM \perp D_0. \end{aligned}$$

Taking $\hat{D} = D \perp \{\xi\}$, we get

$$TM = \hat{D} \perp D^{'}, \qquad \bar{\phi}\hat{D} = \hat{D}.$$

Theorem 3.1. *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then M is totally geodesic if and only if $\bar{g}(Y, A_W X) = \bar{g}(Y, D^l(X,W))$, $\nabla_X \bar{\phi}Y$ has no components in $\bar{\phi}L_1$, $Y \in \Gamma(TM - \{\xi\})$ or X has no components in $\bar{\phi}L_1$.*

Proof. It is well known that the condition for M being totally geodesic is equivalent to the equations given by

$$\bar{g}(h^l(X,Y), E) = 0, \quad E \in \Gamma(RadTM),$$

and

$$\bar{g}(h^s(X,Y), W) = 0, \quad W \in \Gamma(S(TM^{\perp})).$$

In view of (2.7), (2.9) with (2.20), we obtain

$$\begin{aligned} \bar{g}(h^l(X,Y), E) &= \bar{g}(\bar{\nabla}_X Y, E) \\ &= -\bar{g}(\bar{\phi}\bar{\nabla}_X Y, \bar{\phi}E) + \eta(\bar{\nabla}_X Y)\eta(E) \\ &= -\bar{g}(\bar{\phi}\bar{\nabla}_X Y, \bar{\phi}E) \\ &= \bar{g}((\bar{\nabla}_X \bar{\phi})Y, \bar{\phi}E) - \bar{g}(\bar{\nabla}_X \bar{\phi}Y, \bar{\phi}E) \\ &= \bar{g}(X, \bar{\phi}E)\eta(Y) - \bar{g}(\nabla_X \bar{\phi}Y, \bar{\phi}E) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \bar{g}(h^s(X,Y), W) &= \bar{g}(\bar{\nabla}_X Y, W) \\ &= X\bar{g}(Y, W) - \bar{g}(Y, \bar{\nabla}_X W) \\ &= -\bar{g}(Y, \bar{\nabla}_X W) \\ &= \bar{g}(Y, A_W X) - \bar{g}(Y, D^l(X,W)). \end{aligned} \tag{3.4}$$

The proof follows from (3.3) and (3.4). □

Definition 3.2. A paracontact CR-lightlike submanifold of a para-Sasakian manifold is called $\tilde{D}-$geodesic paracontact CR-lightlike submanifold if second fundamental form h satisfies

$$h(X,Y) = 0, \quad \forall X, Y \in \Gamma(\tilde{D}).$$

Theorem 3.2. *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then M is $\tilde{D}-$geodesic if and only if $\nabla^*_X \bar{\phi}E \in \Gamma(\bar{\phi}RadTM \perp \bar{\phi}L_2)$ and $\nabla_X Y$ has no components in $\bar{\phi}L_2$ for any $X, Y \in \Gamma(\tilde{D})$.*

Proof. By using the definition, M is $\tilde{D}-$geodesic if and only if

$$\bar{g}(h^l(X,Y),E)=0, \quad X,Y\in\Gamma(\tilde{D}),\ E\in\Gamma(RadTM)$$

and

$$\bar{g}(h^s(X,Y),W)=0, \quad X,Y\in\Gamma(\tilde{D}),\ W\in\Gamma(S(TM^{\perp})).$$

So, from (2.7), (2.10) with (2.20), we obtain

$$\begin{aligned}\bar{g}(h^l(X,Y),E) &= \bar{g}(\bar{\nabla}_X Y,E)\\ &= -\bar{g}(\bar{\phi}\bar{\nabla}_X Y,\bar{\phi}E)+\eta(\bar{\nabla}_X Y)\eta(E)\\ &= \bar{g}((\bar{\nabla}_X\bar{\phi})Y,\bar{\phi}E)-\bar{g}(\bar{\nabla}_X\bar{\phi}Y,\bar{\phi}E)\\ &= -\bar{g}(\bar{\nabla}_X\bar{\phi}Y,\bar{\phi}E)\\ &= \bar{g}(\bar{\phi}Y,\bar{\nabla}_X\bar{\phi}E)\\ &= \bar{g}(\bar{\phi}Y,\nabla^*_X\bar{\phi}E) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\bar{g}(h^s(X,Y),W) &= \bar{g}(\bar{\nabla}_X Y,W)\\ &= -\bar{g}(\bar{\phi}\bar{\nabla}_X Y,\bar{\phi}W)+\eta(\bar{\nabla}_X Y)\eta(W)\\ &= \bar{g}((\bar{\nabla}_X\bar{\phi})Y,\bar{\phi}W)-\bar{g}(\bar{\nabla}_X\bar{\phi}Y,\bar{\phi}W)\\ &= -\bar{g}(\bar{\nabla}_X\bar{\phi}Y,\bar{\phi}W)\\ &= -\bar{g}(\nabla_X Y,\bar{\phi}W), \end{aligned} \tag{3.6}$$

which completes the proof. □

Definition 3.3. A paracontact CR-lightlike submanifold of a para-Sasakian manifold is called $\breve{D}-$geodesic paracontact CR-lightlike submanifold if its second fundamental form h satisfies

$$h(X,Y)=0, \quad \forall X,Y\in\Gamma(\breve{D}).$$

Theorem 3.3. *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then M is $\breve{D}-$geodesic if and only if $A^*_E X$, $A_W X$ have no components in $\bar{\phi}L_2\perp\bar{\phi}RadTM$ for any $X,Y\in\Gamma(\breve{D})$.*

Proof. We know that M is $\breve{D}-$geodesic if and only if

$$\bar{g}(h^l(X,Y),E)=0, \quad X,Y\in\Gamma(\breve{D}),\ E\in\Gamma(RadTM)$$

and

$$\bar{g}(h^s(X,Y),W)=0, \quad X,Y\in\Gamma(\breve{D}),\ W\in\Gamma(S(TM^{\perp})).$$

So, in view of (2.7) and (2.9), we obtain

$$\begin{aligned}\bar{g}(h^l(X,Y),E) &= \bar{g}(\bar{\nabla}_X Y,E)\\ &= X\bar{g}(Y,E)-\bar{g}(Y,\bar{\nabla}_X E)\\ &= -\bar{g}(Y,\bar{\nabla}_X E)\\ &= -\bar{g}(Y,A^*_E X) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\bar{g}(h^s(X,Y),W) &= \bar{g}(\bar{\nabla}_X Y,W)\\ &= X\bar{g}(Y,W)-\bar{g}(Y,\bar{\nabla}_X W)\\ &= -\bar{g}(Y,\bar{\nabla}_X W)\\ &= \bar{g}(Y,A_X W). \end{aligned} \tag{3.8}$$

The proof follows from (3.7) and (3.8). □

Definition 3.4. A paracontact CR-lightlike submanifold of a para-Sasakian manifold is called mixed geodesic if its second fundamental form h satisfies

$$h(X,Y)=0,$$

for any $X\in\Gamma(\tilde{D})$, $Y\in\Gamma(\breve{D})$.

Theorem 3.4. *Let M be a paracontact CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then M is mixed geodesic if and only if $A_{\bar{\phi}V}U$ has no components in $\bar{\phi}RadTM\perp\bar{\phi}L_2$, for any $U\in\Gamma(\tilde{D})$, $V\in\Gamma(\breve{D})$.*

Proof. Assume that M is mixed geodesic. Therefore we can state

$$\bar{g}(h^l(U,V),E)=0,\ U\in\Gamma(\tilde{D}),\ V\in\Gamma(\breve{D}),\ E\in\Gamma(RadTM)$$

and

$$\bar{g}(h^s(U,V),W)=0,\ U\in\Gamma(\tilde{D}),V\in\Gamma(\breve{D}),\ W\in\Gamma(S(TM^\perp)).$$

By use of (2.7), (2.9) with (2.20), we obtain

$$\begin{aligned}\bar{g}(h^l(U,V),E) &= \bar{g}(\bar{\nabla}_UV,E)\\ &= -\bar{g}(\bar{\phi}\bar{\nabla}_UV,\bar{\phi}E)+\eta(\bar{\nabla}_UV)\eta(E)\\ &= -\bar{g}(\bar{\phi}\bar{\nabla}_UV,\bar{\phi}E)\\ &= \bar{g}((\bar{\nabla}_U\bar{\phi})V,\bar{\phi}E)-\bar{g}(\bar{\nabla}_U\bar{\phi}V,\bar{\phi}E)\\ &= -\bar{g}(\bar{\nabla}_U\bar{\phi}V,\bar{\phi}E)\\ &= \bar{g}(A_{\bar{\phi}V}U,\bar{\phi}E)\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}\bar{g}(h^s(U,V),W) &= \bar{g}(\bar{\nabla}_UV,W)\\ &= -\bar{g}(\bar{\phi}\bar{\nabla}_UV,\bar{\phi}W)+\eta(\bar{\nabla}_UV)\eta(W)\\ &= -\bar{g}(\bar{\phi}\bar{\nabla}_UV,\bar{\phi}W)\\ &= \bar{g}((\bar{\nabla}_U\bar{\phi})V,\bar{\phi}W)-\bar{g}(\bar{\nabla}_U\bar{\phi}V,\bar{\phi}W)\\ &= -\bar{g}(\bar{\nabla}_U\bar{\phi}V,\bar{\phi}W)\\ &= \bar{g}(A_{\bar{\phi}V}U,\bar{\phi}W).\end{aligned}\tag{3.10}$$

The proof follows from (3.9) and (3.10). □

4 Geodesic Paracontact Screen CR-light like Submanifolds

Definition 4.1. Suppose that $(M,g,S(TM),S(TM^\perp))$ is a lightlike submanifold of a para-Sasakian manifold $(\bar{M},\bar{g})$, such that $\xi\in\Gamma(TM)$. If the following conditions are provided on M then M is called a paracontact screen CR-lightlike submanifold;

i) There exist real non-null distributions D and $D^\perp$, such that

$$S(TM)=D\perp D^\perp\perp\{\xi\},\ \ \bar{\phi}(D^\perp)\subset S(TM^\perp),\ \ D\cap D^\perp=\{0\},\tag{4.1}$$

where $D^\perp$ is the orthogonal complementary to $D\perp\{\xi\}$ in $S(TM)$.

ii) $\bar{\phi}(RadTM)=RadTM,\qquad \bar{\phi}(ltr(TM))=ltr(TM),\qquad \bar{\phi}(D)=D.$

Thus one can state following decompositions:

$$\begin{array}{ll} TM & = \bar{D} \perp D^{\perp} \perp \{\xi\}, \\ \bar{D} & = D \perp RadTM. \end{array} \tag{4.2}$$

Let us denote $\mathring{D} = \bar{D} \perp \{\xi\}$. For any X tangent to M, we put

$$\bar{\phi}X = PX + FX, \tag{4.3}$$

where $PX \in \Gamma(\bar{D})$ and $FX \in \Gamma(\bar{\phi}D^{\perp})$.

Similarly, for any $W \in \Gamma(S(TM^{\perp}))$, we put

$$\bar{\phi}W = BW + CW, \tag{4.4}$$

where $BW \in \Gamma(D^{\perp})$ and $CW \in \Gamma(S(TM^{\perp}) - \bar{\phi}D^{\perp})$.

Theorem 4.1. *A paracontact screen CR-lightlike submanifold M of a para-Sasakian manifold is totally geodesic if and only if*

$$(L_E\bar{g})(X,Y) = 0 \quad and \quad (L_W\bar{g})(X,Y) = 0,$$

for any $X, Y \in \Gamma(TM)$, $E \in \Gamma(RadTM)$ and $W \in \Gamma(S(TM^{\perp}))$.

Proof. Assume that M is totally geodesic. Then we get

$$\bar{g}(h(X,Y),E) = 0, \; X \in \Gamma(\bar{D}), Y \in \Gamma(D^{\perp}), E \in \Gamma(RadTM)$$

and

$$\bar{g}(h(X,Y),W) = 0, \quad X \in \Gamma(\bar{D}), \; Y \in \Gamma(D^{\perp}), \; W \in \Gamma(S(TM^{\perp})).$$

By use of property of Lie derivative, we have

$$\begin{array}{rcl} \bar{g}(h^l(X,Y),E) & = & \bar{g}(\bar{\nabla}_X Y, E) \\ & = & X\bar{g}(Y,E) - \bar{g}(Y, \bar{\nabla}_X E) \\ & = & \bar{g}(Y,[E,X]) - \bar{g}(Y, \bar{\nabla}_X E) \\ & = & \bar{g}(Y,[E,X]) - E(\bar{g}(Y,X)) + \bar{g}(\bar{\nabla}_E Y, X) \\ & = & \bar{g}(Y,[E,X]) - E(\bar{g}(Y,X)) + \bar{g}(X,[E,Y]) \\ & & +\bar{g}(\bar{\nabla}_Y E, X) \\ & = & \bar{g}(Y,[E,X]) - E(\bar{g}(Y,X)) + \bar{g}(X,[E,Y]) \\ & & +Y(\bar{g}(E,X)) - \bar{g}(E, \bar{\nabla}_Y X) \\ & = & -\bar{g}(E, \bar{\nabla}_Y X) - (L_E\bar{g})(X,Y) \\ & = & -\bar{g}(h(X,Y),E) - (L_E\bar{g})(X,Y), \end{array}$$

which gives

$$(L_E\bar{g})(X,Y) = -2\bar{g}(h(X,Y),E).$$

Similarly, we get

$$(L_W\bar{g})(X,Y) = -2\bar{g}(h(X,Y),W).$$

Hence the proof is completed. □

Definition 4.2. A paracontact screen CR-lightlike submanifold of a para-Sasakian manifold is called mixed geodesic if its second fundamental form h satisfies

$$h(X,Y) = 0,$$

for any $X \in \Gamma(\bar{D})$, $Y \in \Gamma(D^{\perp})$.

Theorem 4.2. *Let M be a paracontact screen CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then M is mixed geodesic if and only if $\nabla^t_U \bar{\phi} V \in \Gamma(D^\perp)$, $A_{\bar{\phi}V} U \in \Gamma(\mathring{D})$, for any $U \in \Gamma(\mathring{D})$, $V \in \Gamma(D^\perp)$.*

Proof. A paracontact screen CR-lightlike submanifold M is mixed geodesic if and only if

$$h(U, V) = \bar{\nabla}_U V - \nabla_U V = 0.$$

From definition of paracontact screen CR-lightlike submanifold there exists $W \in \Gamma(S(TM^\perp))$ such that

$$\bar{\phi} W = V. \tag{4.5}$$

Thus using (2.23), (4.3) and (4.4), we have

$$\begin{aligned} 0 &= \bar{\nabla}_U \bar{\phi} W - \nabla_U V \\ &= \bar{\phi} \bar{\nabla}_U W - \nabla_U V \\ &= \bar{\phi}(-A_W U + \nabla^t_U W) - \nabla_U V \\ &= -P A_W U - F A_W U + B \nabla^t_U W \\ &\quad + C \nabla^t_U W - \nabla_U V. \end{aligned}$$

In view of (4.3) and (4.4), we know that $F A_W U = 0$ and $C \nabla^t_U W = 0$. So we obtain $\nabla^t_U W \in \Gamma(\bar{\phi} D^\perp)$, $A_W U \in \Gamma(\mathring{D})$. From (4.5) and (2.17) the assertion is proved. □

Theorem 4.3. *Let M be a paracontact screen CR-lightlike submanifold of a para-Sasakian manifold $\bar{M}$. Then $D^\perp$ defines a totally geodesic foliation if and only if $h^s(S, \bar{\phi} N)$ and $h^s(S, \bar{\phi} Z)$ has no components in $\Gamma(\bar{\phi} D^\perp)$, for any $S \in \Gamma(D^\perp)$ and $Z \in \Gamma(\bar{D})$.*

Proof. Assume that, $D^\perp$ is a totally geodesic foliation. In this case we have

$$\nabla_S T \in \Gamma(D^\perp), \qquad \forall S, T \in \Gamma(D^\perp),$$

which gives

$$g(\nabla_S T, Z) = 0, \quad Z \in \Gamma(\bar{D})$$

and

$$g(\nabla_S T, N) = 0, \quad N \in \Gamma(ltr(TM)).$$

Then in view of (2.23), we obtain

$$\begin{aligned} g(\nabla_S T, Z) &= g(\bar{\nabla}_S T, Z) \\ &= S g(T, Z) - g(T, \bar{\nabla}_S Z) \\ &= g(\bar{\phi} T, \bar{\phi} \bar{\nabla}_S Z) - \eta(T) \eta(\bar{\nabla}_S Z) \\ &= g(\bar{\phi} T, \bar{\phi} \bar{\nabla}_S Z) \\ &= -g(\bar{\phi} T, (\bar{\nabla}_S \bar{\phi}) Z) + g(\bar{\phi} T, \bar{\nabla}_S \bar{\phi} Z) \\ &= g(\bar{\phi} T, \bar{\nabla}_S \bar{\phi} Z) \\ &= g(\bar{\phi} T, h^s(S, \bar{\phi} Z)), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} g(\nabla_S T, N) &= g(\bar{\nabla}_S T, N) \\ &= -g(\bar{\phi} \bar{\nabla}_S T, \bar{\phi} N) + \eta(\bar{\nabla}_S T) \eta(N) \\ &= -g(\bar{\phi} \bar{\nabla}_S T, \bar{\phi} N) \\ &= g((\bar{\nabla}_S \bar{\phi}) T, \bar{\phi} N) - g(\bar{\nabla}_S \bar{\phi} T, \bar{\phi} N) \\ &= -g(\bar{\nabla}_S \bar{\phi} T, \bar{\phi} N) \\ &= -S g(\bar{\phi} T, \bar{\phi} N) + g(\bar{\phi} T, \bar{\nabla}_S \bar{\phi} N) \\ &= g(\bar{\phi} T, h^s(S, \bar{\phi} N)). \end{aligned} \tag{4.7}$$

Thus, the proof follows from (4.6) and (4.7). □

Conclusion

Studying in submanifolds of para-Sasakian manifolds, we examine that geodesic paracontact CR-lightlike submanifolds and geodesic paracontact screen CR-lightlike submanifods.

Competing Interests

Authors have declared that no competing interests exist.

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