



A Case of Chromatic-equivalence Implying Tutte-equivalence

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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ABSTRACT

If two non isomorphic graphs are Tutte-equivalent then they are chromatic-equivalent. The opposite is not always true. We give a case of chromatic-equivalence implying Tutte-equivalence.

Keywords: Tutte polynomial; T -equivalence; χ -equivalent; chromatic polynomial.

1 INTRODUCTION

There are many polynomials associated with a graph G . Polynomials play an important role in the study of graphs as they encode various information about a graph. Two of the most studied polynomials for graphs are the Tutte and the chromatic polynomials.

The chromatic polynomial of a graph G , $P(G; \lambda)$, was originally defined by Birkhoff [1], in 1912. Thereafter, Read [2], gave an introduction to the

theory of chromatic polynomials which aroused a lot of interest. This is one of the widely studied polynomials in graph theory: we refer to [3], for further details.

Two graphs G and H are χ -equivalent if and only if $P(G; \lambda) = P(H; \lambda)$. We denote χ -equivalence of G and H by $G \sim H$. It is obvious that if \mathcal{G} is a family of graphs, then \mathcal{G} can be partitioned into χ -equivalent classes. The χ -equivalence class determined by a graph G will be denoted by $[G]$. A graph G is χ -unique if and only if $[G] = \{G\}$.

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In his introductory paper [2], Read posed a few questions which initiated the research in chromatic polynomials. One of the questions he posed is, what is a necessary and sufficient condition for two non-isomorphic graphs to be χ -equivalent? In particular, he gave a less trivial pair of non-isomorphic graphs G_1 and G_2 shown in Fig. 1 with the same chromatic polynomial.

In response to this question, a lot of literature has been gathered on construction of χ -equivalent graphs and χ -unique graphs, for recent research, see, [4], [5], [6], and for a summary and directions of the literature collected, we refer to [3]. Read's question is still not fully settled. In addition, this paper was motivated by the suggestion made by Wilf [7], of a further study on the relationship between closed subsets and the computation of the chromatic polynomial.

In 1954, Tutte introduced a two variable polynomial for a graph and called it the dichromate of the graph. This polynomial is

now called the Tutte polynomial of a graph and is denoted by $T(G; x, y)$. This polynomial is of central importance in graph theory as it carries a lot of information about a graph. In addition the widely studied chromatic polynomial is simply an evaluation of the Tutte polynomial $(-1)^{|V(G)|-\omega(G)}\lambda^C T(G; 1-\lambda, 0)$ where $\omega(G)$ is the number of connected components of G . Two graphs G and H are T -equivalent if and only if $T(G; x, y) = T(H; x, y)$. We denote T -equivalence of G and H by $G \stackrel{T}{\sim} H$. A graph G is T -unique if every graph H that is T -equivalent to G is isomorphic to G . We refer the reader to [8], [9], [10], for some literature on T -equivalence and T -unique.

It is obvious that if two non isomorphic graphs are T -equivalent then they are χ -equivalent since $\chi(G; \lambda) = (-1)^{|V(G)|-\omega(G)}T(G; 1-\lambda, 0)$. The opposite is not always true. For example the pair of non-isomorphic graphs G_1 and G_2 shown in Fig. 1 share the same chromatic polynomial but has the following Tutte polynomials:

$$T(G_1; x, y) = 18xy + 18xy^2 + 4xy^3 + 18x^2y + 2x^2y^2 + 4x^3y + 4y + 11y^2 + 11y^3 + 5y^4 + y^5 + x^5 + 5x^4 + 11x^3 + 11x^2 + 4x$$

$$T(G_2; x, y) = 16xy + 15xy^2 + 5xy^3 + xy^4 + 16x^2y + 4x^2y^2 + 4x^3y + 4y + 9y^2 + 7y^3 + 3y^4 + y^5 + x^5 + 5x^4 + 11x^3 + 11x^2 + 4x$$

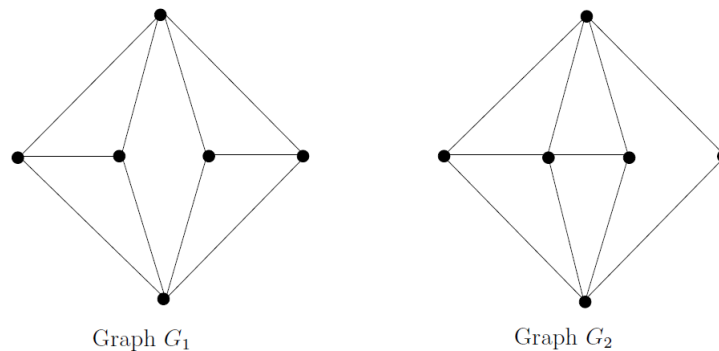


Fig. 1

In this paper, we begin by reviewing the theory of coboundary polynomials and k -defect polynomials relevant to this paper. Then we give the relationship between closed subsets and the computation of the chromatic polynomial. Finally, we state and prove the main result of this paper and give an example of a set of non-isomorphic graphs which satisfy the main theorem.

2 THE THEORY OF CO-BOUNDARY POLYNOMIALS

A *coloring* of a graph G is an assignment of colors to each vertex of G . A coloring of G in which adjacent vertices are not allowed to have the same color is called a *proper coloring*. The *chromatic polynomial* of a graph G , $P(G; \lambda)$, expresses the number of different proper coloring of G with at most λ colors. Moreover, the characteristic polynomial of a graph G , $\chi(G; \lambda) = \frac{P(G; \lambda)}{\lambda^{\omega(G)}}$ where $\omega(G)$ is the number of components of G . A coloring of G in which vertices are allowed to be adjacent to other vertices with the same color as themselves is called a *bad coloring*. An edge is called *bad* if it joins two vertices of the same color in a bad coloring. The *k-defect polynomial*, $\phi_k(G; \lambda)$, counts the number of ways of coloring G with at most λ colors and having exactly k bad edges. If $\phi_0(G; \lambda)$ counts the number of ways of colouring G with λ colours and having exactly 0 bad edges, then $P(G; \lambda) = \phi_0(G; \lambda)$. For further details of the theory of *k-defect* polynomials: we refer the reader to [11, 12]. It is clear that if G is an edgeless graph, then $\chi(G; \lambda) = 1$.

Recall that the rank of a graph G is $r(G) = |V| - \omega(G)$. The *coboundary polynomial* of a graph G , with an edge set E , is a polynomial in two independent variables λ and s , denoted by $B(G; \lambda, s)$ and is defined as

$$B(G; \lambda, s) = \sum_{X \subseteq E} (s-1)^{|X|} \lambda^{r(G)-r(X)}.$$

From this definition it follows that $B(G; \lambda, 0) = \chi(G; \lambda)$. It is indicated in [11] that the coboundary polynomial of a graph G is equal to the coboundary polynomial of its cycle matroid $M(G)$, thus $B(G; \lambda, s) = B(M(G); \lambda, s)$. Thus the theory of coboundary polynomials for matroids generalizes for graphs.

The coboundary polynomial of a graph G was defined and studied by Crapo [13], as a generating function in s given by

$$B(G; \lambda, s) = \sum_{k=0}^{|E|} s^k \phi_k(G; \lambda) \quad (2.1)$$

where $\phi_k(G; \lambda)$ is a polynomial in λ called the

k-defect polynomial of a graph. The following proposition summarizes some of the well known relationships of the coboundary polynomial, Tutte polynomial and the chromatic polynomial of graphs relevant to this paper.

Proposition 2.1. *Let G be a graph of rank- r , $B(G; \lambda, s)$ its coboundary polynomial and $T(G; x, y)$ its Tutte polynomial, then*

$$B(G; \lambda, s) = (s-1)^r T(G; \frac{s+\lambda-1}{s-1}, s)$$

or vice versa

$$T(G; x, y) = \frac{1}{(y-1)^r} B(G; (x-1)(y-1), y).$$

Corollary 2.1. *Let G be a rank- r graph. Then*

$$\chi(G; \lambda) = (-1)^r T(G; 1-\lambda, 0).$$

Thus by generalizing the recursion for characteristic polynomial of a graph, see [11], the coboundary polynomial can be computed recursively as given in the next proposition.

Proposition 2.2. *Let G be a graph and let $e \in E$.*

[(Bi)] *If e is neither a loop or an isthmus of G , then*

$$B(G; \lambda, s) = B(G \setminus e; \lambda, s) + (s-1)B(G/e; \lambda, s).$$

[(Bii)] *If e is a loop, then*

$$B(G; \lambda, s) = sB(G \setminus e; \lambda, s).$$

[(Biii)] *If e is an isthmus, then*

$$B(G; \lambda, s) = (s + \lambda - 1)B(G/e; \lambda, s).$$

The following proposition is derived from Equation 2.1, see [11].

Proposition 2.3. *Let G be a graph and $\mathcal{L}(G)$ the set of all closed sets of G . Then*

$$B(G; \lambda, s) = \sum_{X \in \mathcal{L}(G)} s^{|X|} \chi(G/X; \lambda).$$

The following proposition is an immediate consequence of Equation 2.1 and Proposition 2.3.

Proposition 2.4. *Let G be a graph. Then*

$$\phi_k(G; \lambda) = \sum_{X \in \mathcal{L}(G), |X|=k} \chi(G/X; \lambda).$$

3 A CONDITION FOR EQUIVALENCE

In this section we give the computation of the chromatic polynomial in terms of closed sets, state and prove the main result of this paper. In addition we conclude by giving an example of graphs which satisfy the main theorem. We need some lemmas and propositions before stating and proving the main result.

Lemma 3.1. *Let G be a rank- r graph. Then $B(G; \lambda, 1) = \lambda^r$.*

Proof: This is clear by applying the deletion and contraction method. Note that all the minors obtained by contracting contribute a zero. So we only need to delete one element at a time and apply Proposition 2.2(iii) once we have a coloop to get the result.

Lemma 3.2. *Let G be a rank- r graph. Then*

$$\chi(G, \lambda) = \lambda^r - \sum_{k=1}^{|E|} \phi_k(G; \lambda).$$

Proof: Recall that $\chi(G; \lambda) = \phi_0(G; \lambda)$ from the definition. By applying Equation 2.1 we get

$$\begin{aligned} \chi(G, \lambda) &= \lambda^r - \sum_{k=1}^{|E|} \phi_k(G; \lambda) \text{ by Lemma 3.1} \\ &= \lambda^r - \sum_{k=1}^{|E|} \sum_{X \in \mathcal{L}(G), |X|=k} \chi(G/X; \lambda) \text{ by Proposition 2.4} \\ &= \lambda^r - \sum_{i=1}^q \chi(g_i; \lambda) \\ &= \lambda^r - \sum_{i=1}^q \chi(h_i; \lambda) \\ &= \chi(H, \lambda). \end{aligned}$$

Hence G is χ -equivalent to H .

χ - $B(G; \lambda, 1) = \sum_{k=0}^{|E|} \phi_k(G; \lambda)$. Hence

$$\begin{aligned} \phi_0(G; \lambda) &= B(G; \lambda, 1) - \sum_{k=1}^{|E|} \phi_k(G; \lambda) \\ &= \lambda^r - \sum_{k=1}^{|E|} \phi_k(G; \lambda) \text{ by Lemma 3.1.} \end{aligned}$$

The following notation is required to state the main theorem. Let $\mathcal{L}(G)$ be the lattice of closed sets of the graph G . We define the multiset of all minors obtained by contracting closed sets of G to be $Cl(G) = \{G/X, \forall X \in \mathcal{L}(G)\}$. From this notation it is clear that $|\mathcal{L}(G)| = |Cl(G)|$. Recall from section 1 that we denote χ -equivalence of G and H by $G \approx H$. Recall that two graphs G and H are isomorphic up to parallel class if the simplification of G is equal to the simplification of H .

Theorem 3.1. *Let G and H be rank- r graphs, $g_i \in Cl(G)$ and $h_i \in Cl(H)$. Then G is χ -equivalent to H if there exist a bijection ζ from $Cl(G)$ to $Cl(H)$ defined by $\zeta(g_i) = h_i$ if g_i is isomorphic to h_i up to parallel class.*

Proof: A bijection from $Cl(G)$ to $Cl(H)$ implies $|Cl(G)| = |Cl(H)| = q$. By definition of bijection ζ , g_i is isomorphic to h_i up to parallel class which implies that $g_i \approx h_i$. Since ζ is a bijection then for each $i \in \{1, 2, \dots, q\}$ we get $\chi(g_i; \lambda) = \chi(h_i; \lambda)$. Therefore

We define the set of all minors obtained by contracting closed sets of size k of G to be $Cl_k(G) = \{G/X, \forall X \in \mathcal{L}(G) \text{ such that } |X| = k\}$. Recall from section 1 that we denote T -equivalence of G and H by $G \stackrel{T}{\sim} H$.

Theorem 3.2. *Let G and H be rank- r graphs, $g_{i_k} \in Cl_k(G)$ and $h_{i_k} \in Cl_k(H)$ where $k = \{1, 2, \dots, |E|\}$. If G is χ -equivalent to H such that there exist a bijection ζ from $Cl_k(G)$ to $Cl_k(H)$ defined by $\zeta(g_{i_k}) = h_{i_k}$ if g_{i_k} is isomorphic to h_{i_k} up to parallel class then G is T -equivalent to H .*

Proof: A bijection from $Cl_k(G)$ to $Cl_k(H)$ implies $|Cl_k(G)| = |Cl_k(H)|$. Isomorphism up to parallel class implies $\chi(g_{i_k}; \lambda) = \chi(h_{i_k}; \lambda)$. Hence by Proposition 2.4, $\phi_k(G; \lambda) = \phi_k(H; \lambda)$. Therefore, by Equation 2.1 it follows that

$$B(G; \lambda, s) = B(H; \lambda, s). \quad (3.1)$$

Applying Proposition 2.1, and Equation 3.1 we

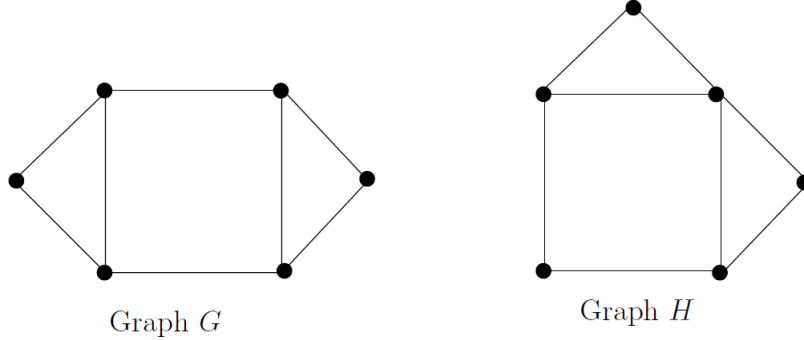


Fig. 2

G and H are χ -equivalent since

$$\begin{aligned} \chi(G; \lambda) &= \lambda^6 - 8\lambda^5 + 26\lambda^4 - 43\lambda^3 + 36\lambda^2 - 12\lambda \\ &= \chi(H; \lambda). \end{aligned}$$

G and H are T -equivalent since

$$\begin{aligned} T(G; x, y) &= x^5 + 3x^4 + 4x^3 + 3x^2 + x + y^3 + 2y^2 + y + 4xy + 5x^2y + 2x^3y + 3xy^2 \\ &= T(H; x, y). \end{aligned}$$

Now, we need to show that there exist a bijection ζ from $Cl_k(G)$ to $Cl_k(H)$ defined by $\zeta(g_{i_k}) = h_{i_k}$ if $g_{i_k} \stackrel{\sim}{\sim} h_{i_k}$. The graphs given in Fig. 3 are simplifications of some of the minors obtained by contracting closed sets in graphs G and H . Recall that S_n , P_n and C_n is a star graph, a path graph and a cycle graph on n vertices, respectively.

have

$$\begin{aligned} T(G; x, y) &= \frac{1}{(y-1)^r} B(G; (x-1)(y-1), y) \\ &= \frac{1}{(y-1)^r} B(H; (x-1)(y-1), y) \\ &= T(H; x, y) \end{aligned}$$

Hence G is T -equivalent to H .

The following theorem is a slight variation of Theorem 3.2.

Theorem 3.3. *Let G and H be rank- r graphs, $g_{i_k} \in Cl_k(G)$ and $h_{i_k} \in Cl_k(H)$ where $k = \{1, 2, \dots, |E|\}$. If G is χ -equivalent to H such that there exist a bijection ζ from $Cl_k(G)$ to $Cl_k(H)$ defined by $\zeta(g_{i_k}) = h_{i_k}$ if $g_{i_k} \stackrel{\sim}{\sim} h_{i_k}$, then G is T -equivalent to H .*

We give an example of a pair of graphs that satisfy Theorem 3.3.

Example 3.1. *Let G and H be the graphs shown in Fig. 2.*

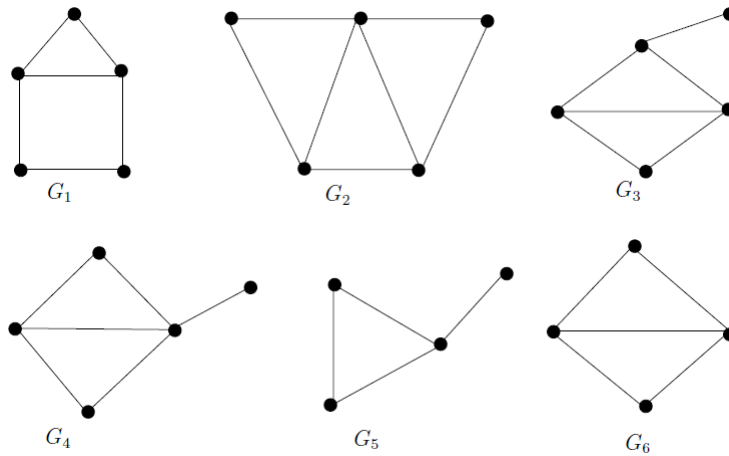


Fig. 3

Table 1

k	$Cl_k(G)$	$Cl_k(H)$
1	$\underbrace{\{G_1, \dots, G_1, G_2, G_2, G_3, G_3\}}_4$	$\underbrace{\{G_1, \dots, G_1, G_2, G_2, G_4, G_4\}}_4$
2	$\underbrace{\{G_6, \dots, G_6, G_5, \dots, G_5, C_4, \dots, C_4, S_4\}}_9$	$\underbrace{\{G_6, \dots, G_6, G_5, \dots, G_5, C_4, \dots, C_4, S_4\}}_9$
3	$\underbrace{\{G_6, G_6, C_3, \dots, C_3, P_3, \dots, P_3\}}_{12}$	$\underbrace{\{G_6, G_6, C_3, \dots, C_3, P_3, \dots, P_3\}}_{12}$
4	$\underbrace{\{P_2, \dots, P_2, P_3, \dots, P_3, C_3, \dots, C_3\}}_8$	$\underbrace{\{P_2, \dots, P_2, P_3, \dots, P_3, C_3, \dots, C_3\}}_8$
5	$\underbrace{\{P_2, \dots, P_2\}}_8$	$\underbrace{\{P_2, \dots, P_2\}}_8$
6	$\underbrace{\{P_2, \dots, P_2\}}_8$	$\underbrace{\{P_2, \dots, P_2\}}_8$
7	$\underbrace{\{\}}_8$	$\underbrace{\{\}}_8$
8	$\{K_1\}$	$\{K_1\}$

Recall the notation $Cl_k(G) = \{G/X, \forall X \in \mathcal{L}(G) \text{ such that } |X| = k\}$. The table below indicates the multisets $Cl_k(G)$ and $Cl_k(H)$ for $k = \{1, 2, \dots, 8\}$ for comparison. From the table, it is clear that the multisets $Cl_k(G) = Cl_k(H)$ for $k = \{2, \dots, 8\}$, hence the bijection ζ is trivial. We only need to show this bijection for $k = 1$.

$$\begin{aligned} \chi(G_3; \lambda) &= (\lambda - 1)\chi(G_6; \lambda) \\ &= \chi(G_4; \lambda). \end{aligned}$$

Hence, there exist a bijection ζ from $Cl_1(G)$ to

$Cl_1(H)$ defined by $\zeta(G_i) = H_i$ where $H_1 = G_1$, $H_2 = G_2$ and $H_3 = G_4$ and $G_i \simeq H_i$.

4 CONCLUSION

Note that the pair of χ -equivalent graphs G_1 and G_2 given in Fig. 1 do not satisfy Theorem 3.2 or Theorem 3.3 and we have shown in the introduction that these graphs G_1 and G_2 are not T -equivalent. To conclude this paper, we

pose the following question, what is a necessary and sufficient condition for two non-isomorphic χ -equivalent graphs to be T -equivalent?

Competing Interests

Author has declared that no competing interests exist.

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