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Estimation of Thermal Pollution Using Numerical Simulation of Energy Equation Coupled with Viscous Burgers' Equation

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Abstract

In this paper, we implement energy equation coupled with viscous Burgers' equation as a mathematical model for the estimation of thermal pollution of river water. The model is a nonlinear system of partial differential equations (PDEs) that read as an initial and boundary value problem (IBVP). For the numerical solution of the IBVP, we investigate an explicit second-order Lax-Wendroff type scheme for nonlinear parabolic PDEs. We present the numerical solutions graphically as a temperature profile, which shows good qualitative agreement with natural phenomena of heat transfer. We estimate the thermal pollution of water caused by industrialization on the bank of a river.

Keywords

Viscous Burgers' Equation, Energy Equation, Heat Transfer, Lax-Wendroff Type Scheme, Thermal Pollution

1. Introduction

We study the energy equation together with the viscous Burgers' equation. The energy equation can reveal the thermal pollution in river water [1]. We consider the nonlinear system of equations as a model equation and investigate numerical solutions with the initial conditions and boundary conditions to make us understand the thermal pollution distribution through an open medium like rivers, lakes, and water sources [2]. Thermal pollution occurs with an increase in water temperatures in any water source. This pollution in water caused due to human activities such as electric power plants, steel melting factories, and boilers from industries that release a large amount of heat water [3]. The water temperature increases the decline of oxygen content in water. These cause difficulty for aquatic life, which cannot tolerate high temperatures [4]. Most sponges, mollusks, and shrimps die at temperatures above 27 degrees Celsius. In addition, the energy equation analyses the flow in an autoclave for curing aerospace parts using an incompressible flow assumption. It has wide applications in other disciplines in a liquid rocket engine, liquid fuel, and oxidizer used as a coolant in various components such as the bearing in the oxidizer and fuel turbopump. Also, the flow in an autoclave for curing aerospace parts can be analyzed using an incompressible flow assumption. It is widely involved in generating power from conventional fossil fuels, nuclear, and geothermal energy sources [5] [6].

Several numerical methods are presented in the literature to solve Burgers' equation and energy equation. Agusta and Bamingbola [7] surveyed the numerical treatment of the mathematical model for water pollution using the implicit centred difference in space and a forward difference method in time for the evaluation of the generalized transport equation. Changiun Zhu, Liping Wa and Sha Li [8] made a numerical simulation of river water pollution by using the grey differential model. Jima, Shiferaw and Tsegaye [9] solved coupled viscous Burgers' equation with appropriate initial and boundary conditions applying the differential quadrature method based on Fourier expansion basis. A numerical method was investigated by P.D. Lax and B. Wendroff [10] for the solution to the system of hyperbolic conservation laws. Jiequan Li [11] investigated Lax-Wendroff type scheme for hyperbolic problems. We use an explicit second-order Lax-Wendroff type scheme [12] for numerical solutions of energy equation and viscous Burgers' equation, which are nonlinear parabolic PDEs. Here the first-order terms are discretized in second-order same as Lax-Wendroff scheme of hyperbolic PDE.

In Section 2, we present the governing equation to analyse thermal pollution. In Section 3, we apply Lax-Wendroff type scheme to obtain a numerical solution of Viscous Burgers' equation and energy equation respectively as an IBVP using Neumann boundary conditions [13] [14]. In Section 4, we implement the numerical scheme for the governing equations to simulate thermal pollution in water with respect to different times and discrete positions in space.

2. The Governing Equations

We investigate the thermal pollution in water using well-known energy equation and viscous Burgers' equation. The system of PDEs is as follows

$$\begin{gathered} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \\ \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2} \end{gathered} \tag{1}$$

where, $u\frac{\partial u}{\partial x}$ and $u\frac{\partial T}{\partial x}$ are convection terms and $\nu\frac{\partial^2 u}{\partial x^2}$ is diffusion term for viscous Burgers' equation with kinematic viscous coefficient ν and $\alpha\frac{\partial^2 T}{\partial x^2}$ is diffusion term for energy equation with heat diffusion co-efficient α. $u(x,t)$ and $T(x,t)$ are the velocity of flow the water temperature respectively.

Energy equation is a non-linear equation and its analytic solution is not yet available. We consider the model equation as an IBVP for numerical solution by setting the initial conditions $u(x,0)=u_0(x)$ and $T(x,0)=T_0(x)$, left boundary conditions $u(x_L,t)=u_L(t)$ and $T(x_L,t)=T_L(t)$. Neumann condition is considered for right boundary. Burgers' equation provides the velocity of the fluid flow and the energy equation present temperature profile of water by which we interpret thermal pollution in water.

3. Numerical Method for Governing Equations

In this section, we discuss the numerical scheme for Equation (1). A second-order discretization for viscous Burgers' equation and energy equation is performed, which is known as Lax-Wendroff type scheme [12]. For second-order Lax-Wendroff type scheme of the viscous Burgers' equation, we discretize the inviscid part of the equation in half-time step Lax-Friedrich scheme then substitute those values in half-time step Leapfrog scheme and combine them with second-order central discretized viscid part of Burgers' equation. Again, for second-order Lax-Wendroff type scheme [12] of energy equation, we discretize the convective part of the equation in half-time step Lax-Friedrich scheme then substitute those values in half-time step Leapfrog scheme and combine with second-order central discretized diffusive part of energy equation.

3.1. Lax-Wendroff Type Scheme for Burgers' Equation

We take forward discretization in time derivative and first order space derivative, for Lax-Friedrich scheme and the discrete equation for inviscid part of Burgers' equation

$$u_j^{n+1}=\frac{1}{2}\left(u_{j+1}^n+u_{j-1}^n\right)-\frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(u_{j+1}^n-u_j^n\right) \tag{2}$$

which is Lax-Friedrich scheme. Now take half-time step for (2) and we have

$$u_{j+\frac{1}{2}}^{n+\frac{1}{2}}=\frac{1}{2}\left(u_{j+1}^n+u_j^n\right)-\frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(u_{j+1}^n-u_j^n\right) \tag{3}$$

and

$$u_{j-\frac{1}{2}}^{n+\frac{1}{2}}=\frac{1}{2}\left(u_j^n+u_{j-1}^n\right)-\frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(u_j^n-u_{j-1}^n\right) \tag{4}$$

Half-time step Leapfrog scheme for inviscid part of Burgers' equation is

$$u_j^{n+1}=u_j^n-\frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(u_{j+\frac{1}{2}}^{n+\frac{1}{2}}-u_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) \tag{5}$$

Therefore, the discrete form of viscous Burgers' equation comes from (5) and the central difference of second-order space derivative. Then the discrete equation reads as

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) + \nu * \Delta t\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\left(\Delta x\right)^2}\right) \tag{6}$$

Now substituting the values of $u_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ and $u_{j-\frac{1}{2}}^{n+\frac{1}{2}}$ in (6), we have

$$\begin{aligned} u_j^{n+1} = u_j^n &- \frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(\frac{1}{2}\left(u_{j+1}^n + u_j^n\right) - \frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(u_{j+1}^n - u_j^n\right) - \frac{1}{2}\left(u_j^n + u_{j-1}^n\right)\right. \\ &\left. - \frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(u_j^n - u_{j-1}^n\right)\right) + \frac{\nu * \Delta t}{\left(\Delta x\right)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \end{aligned} \tag{7}$$

which the Formula (7) is the Lax-Wendroff type scheme for viscous Burgers' equation.

3.2. Lax-Wendroff Type Scheme for Energy Equation

We take forward discretization in time derivative and first order space derivative, for Lax-Friedrich scheme and the discrete equation for convective part of energy equation

$$T_j^{n+1} = \frac{1}{2}\left(T_{j+1}^n + T_{j-1}^n\right) - \frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(T_{j+1}^n - T_j^n\right) \tag{8}$$

which is Lax-Friedrich scheme. Take half-time step for (8) and we get

$$T_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(T_{j+1}^n + T_j^n\right) - \frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(T_{j+1}^n - T_j^n\right) \tag{9}$$

and

$$T_{j-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(T_j^n + T_{j-1}^n\right) - \frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(T_j^n - T_{j-1}^n\right) \tag{10}$$

Half-time step Leapfrog scheme for convective part of the energy equation is

$$T_j^{n+1} = T_j^n - \frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(T_{j+\frac{1}{2}}^{n+\frac{1}{2}} - T_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) \tag{11}$$

Therefore, the discrete form of energy equation comes from (11) and the central difference of second-order space derivative. Then the discrete equation reads as

$$T_j^{n+1} = T_j^n - \frac{\Delta t}{\Delta x}\left(u_j^n\right)\left(T_{j+\frac{1}{2}}^{n+\frac{1}{2}} - T_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) + \alpha * \Delta t\left(\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\left(\Delta x\right)^2}\right) \tag{12}$$

Now substituting the values of $T_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ and $T_{j-\frac{1}{2}}^{n+\frac{1}{2}}$ in (12), we have

$$\begin{aligned} T_j^{n+1} = T_j^n &- \frac{\Delta t}{2\Delta x}\left(u_j^n\right)\left(T_{j+1}^n - T_{j-1}^n\right) + \left(\frac{\Delta t}{\Delta x}\left(u_j^n\right)\right)^2\left(T_{j+1}^n - 2T_j^n + T_{j-1}^n\right) \\ &+ \frac{\nu * \Delta t}{\left(\Delta x\right)^2}\left(T_{j+1}^n - 2T_j^n + T_{j-1}^n\right) \end{aligned} \tag{13}$$

which the Formula (13) is the Lax-Wendroff scheme for energy equation.

4. Results and Discussions

We apply the numerical scheme to estimate the thermal pollution in a river at different times and individual points of water bodies. The computational result verifies the qualitative behavior of the solution of the energy equation for varying of the parameters. We consider initial flow velocity $u_0(x) = 0.05 * \sin(-0.05x) + 0.05$ with velocity in left boundary $u_L(t) = 0.05$ m·s^{-1} and the viscosity of water flow is considered υ = 0.05 m^2/s. The initial temperature in river water is considered $T_0(x) = 0.05 * \exp(-0.05x) + 20$; the temperature in left boundary is $T_L(t) = 40^{\circ}\text{C}$. Both right boundaries are generated by Neumann boundary conditions. The heat diffusivity in water is α = 0.06 m^2/s. The total execution domain is 100 m and simulation time is 20 minutes. The results are shown in following figures.

From **Figure 1**, we observe that at time t = 1 min. the thermal pollution is not clear, at time t = 5 min., the temperature is transferred, increased with time and the spread out of thermal pollution is noticeable; at time t = 20 min, the water temperature is increased all along the boundary of the river. We notice, thermal pollution is moving to boundary with time and increasing the pollution rate. **Figure 2** shows the graphical scenario of thermal pollution in water after t = 1 hour, which is much alarming for bio-diversity.

In **Figure 3**, the temperature profile has shown with respect to space. The curved lines in the above figure shows the change of temperature at x = 20, 40, 60, 80 and 100 meters. From observation we can say, the thermal pollution in river water is increasing with distance.

In **Figure 4**, we consider different viscous coefficients υ = 0.005, 0.05 and 0.14 m^2/s for fixed heat diffusion rate α = 0.06 m^2/s. From figure we notice that as the viscous coefficient increases the thermal pollution increases along the boundary, which reflects the well-known natural phenomena of viscosity in water.

In **Figure 5**, we consider different heat diffusion rates α = 0.03, 0.06 and 0.14 m^2/s for fixed viscous coefficient υ = 0.05 m^2/s. We observe, for higher heat

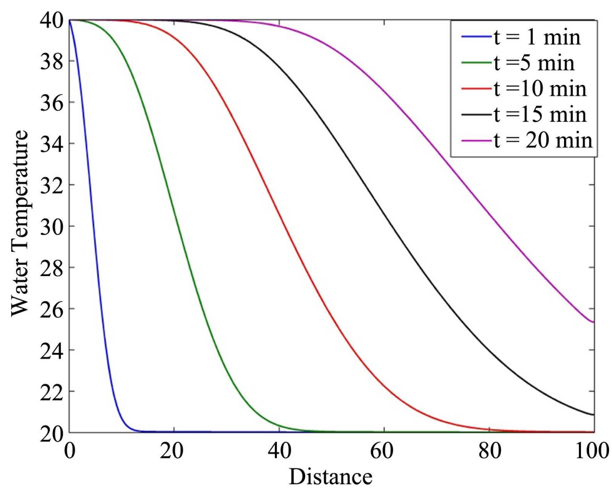

Figure 1. Temperature profile for Lax-Wendroff type Scheme at different time.

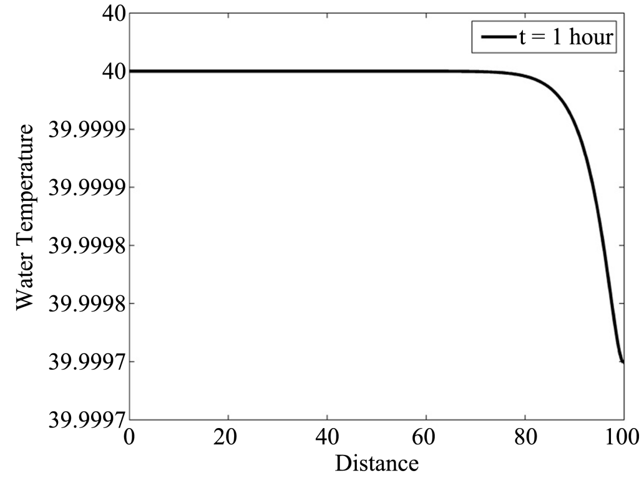

Figure 2. Temperature profile for Lax-Wendroff type Scheme at $t = 1$ hour.

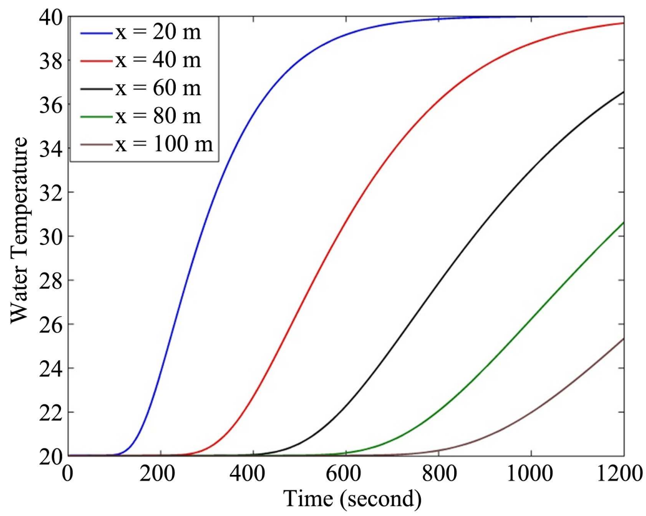

Figure 3. Temperature profile for Lax-Wendroff type scheme at different position.

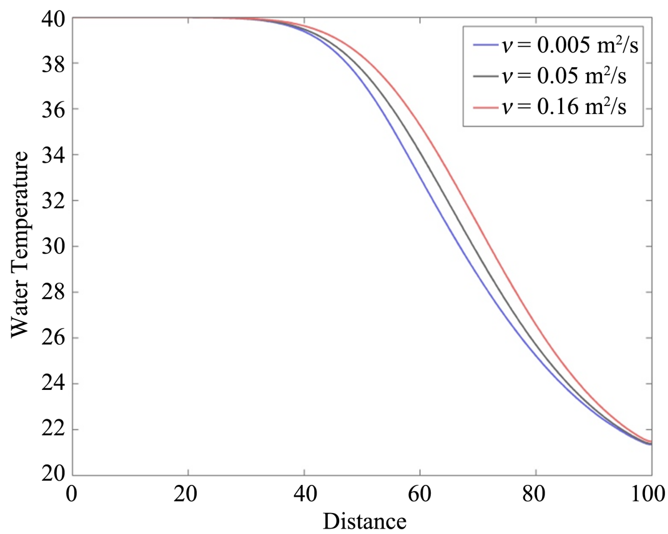

Figure 4. Temperature distribution for different viscous co-efficients at $\alpha = 0.06\ m^2/s$.

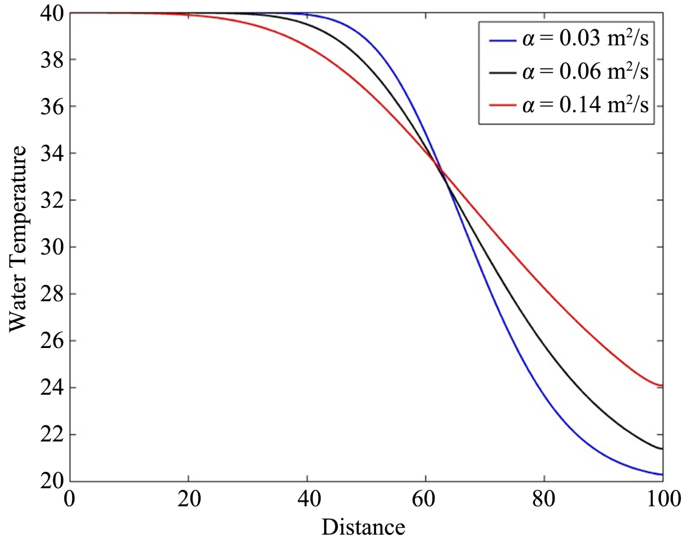

Figure 5. Temperature distribution at different heat diffusion co-efficients at υ = 0.05 m^2/s.

diffusion rate temperature spreads out, as α increases thermal pollution increases with respect to time along the boundary.

From the above discussion, it can be estimate that how far the effect of thermal pollution exists in a river and can idea about the region of river, where the bio-diversity being affected.

5. Conclusion

In this paper, we have discussed the energy equation coupled with the viscous Burgers' equation as a model to estimate the thermal pollution in river water. We have investigated the numerical solution for the Lax-Wendroff type scheme, which is an explicit second-order scheme for nonlinear parabolic PDEs. We have done the discretization of first-order terms of energy equation and viscous Burgers' equation are in second-order which same as the Lax-Wendroff type scheme of hyperbolic PDE. We have presented the numerical solutions qualitatively by varying the values of the viscous and heat diffusion coefficients. The representations showed a good qualitative agreement with the well-known behavior of the solution of the energy equation. The results have shown that the thermal pollution in water is spreading as we have been varying the convection and thermal diffusivity with respect to time and space. The profile of thermal pollution in water has been observed at different times, points of space, viscosity, and heat diffusivity. In the estimation of thermal pollution in river water, the second-order Lax-Wendroff type scheme has shown good qualitative agreement with natural phenomena of heat transfer.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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