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On Nil and Nilpotent Rings and Modules

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Abstract

Throughout the study, all rings are associative with identity and all modules are unitary right *R*-modules. Let *M* be a right *R*-module and $S = End_R(M)$, its endomorphism ring. A submodule *X* of a right *R*-module *M* is called a nilpotent submodule of *M* if I_X is a right nilpotent ideal of *S* and *X* be a nil submodule of *M* if I_X is a right nil ideal of *S*. By definition, a nilpotent submodule is a nil submodule. It is seen that *X* is a fully invariant nilpotent submodule of *M* if and only if I_X is a two-sided nilpotent ideal of *S*. In this paper, we present some results of nil and nilpotent submodules over associative endomorphism rings.

Keywords

Modules, Nilpotent, Radical, Semi Primes

1. Introduction and Preliminaries

Ring theory is an important part of algebra. It has been widely used in Electrical and computer Engineering [1]. Historically, some of the major discoveries in ring theory have helped shape the course of development of modern abstract algebra. Modern ring theory begins when Wedderburn in 1907 proved his celebrated classification theorem for finite dimensional semi-simple algebras over fields. Twenty years later, E. Noether and E. Artin introduced the ascending chain condition and descending chain condition as substitutes for finite dimensionality.

We know that Module theory appeared as a generalization of theory of vector spaces over a field. Every field is a ring and every ring may be considered as a module. Köthe [2] first introduced and investigated the notion of nil ideals in commutative ring theory. Amitsur [3] investigated radicals of polynomial rings. It is important to ascertain when nil and Jacobson radical coincide. It is known that nil rings are Jacobson radicals. Again Jacobson radical of a finitely generated algebra over a field is nil [3]. There were some historical notes on nil ideals and nil radicals due to Amitsur [4]. Radicals of graded rings were introduced and investigated by Jespers *et al.* [5]. There is another notion of radicals in nil and Jacobson radicals in graded rings due to Smoktunowicz [6]. Puczylowski ([7], [8]) investigated some results concerning radicals of associative rings related to Köthe's nil ideal problems. Following [9], if X is a prime submodule of a right R-module M, then the set I_x is a prime ideal of the endomorphism ring S and the converse is true if M is a self-generator. Ali [10] investigated the idempotent and nilpotent submodules of multiplicative modules. Chebotar *et al.* [11] and Klein [12] investigated some results concerning nil ideals of associative rings which do not necessarily have identities. Sanh *et al.* ([13] [14] [15]) introduced the notion of fully invariant submodules and characterized their properties.

2. Nil and Nilpotent Rings

Let R be a ring with DCC on right ideals. Let $\{I_\alpha\}$ be the collection of all nilpotent right ideals of R. Then $N=\sum I_\alpha$ is called the radical of R.

It was shown in [16] that every one-sided or two-sided nilpotent ideal is a nil ideal and the sum of two nilpotent right, left or two-sided ideals is again nilpotent. Using these results we can prove the following corollary over associative arbitrary rings.

Corollary 2.1: Let R be a right noetherian ring. Then each nil one-sided ideal of R is nilpotent.

Proof. Let S be the sum of all the nilpotent right ideals of R. The S is an ideal. Since R is right noetherian, S is the sum of a finite number of nilpotent right ideals and hence S is nilpotent. It follows that the quotient R/S has no nonzero nilpotent right ideals. Let I be a nil one-sided ideal of R. Then the image of I in R/S is zero. Hence $I\subseteq S$.

Again the following propositions give us the property similar to that of rings.

Proposition 2.2 ([17]): If R is a semisimple ring then it has no two-sided ideals except zero and R.

Proposition 2.3: Let R be a semiprime ring with the ACC for right annihilators. Then R has no nonzero nil one-sided ideals.

Proof. Let I be a nonzero one-sided ideal of R and let $0\neq a\in I$ with $r_R(a)$ as large as possible. Since R is semiprime, there is an element $x\in R$ such that $axa\neq 0$. Thus axa is a nonzero element of I such that $r_R(a)\subseteq r_R(axa)$. So $r_R(a)=r_R(axa)$. We have $ax\neq 0$, *i.e.*, $x\notin r_R(a)$. Thus $x\notin r_R(axa)$. So, $(ax)^2\neq 0$. Hence $xax\notin r_R(a)$ implying that $(ax)^3\neq 0$. Therefore, ax and hence, also xa is not nilpotent and $ax\in I$ or $xa\in I$.

Definition 2.4: The nil radical of a ring R is defined to be the radical ideal with respect to the property that "a two-sided ideal is nil" and is denoted by $N(R)$. That is, $N(R)$ is the largest two-sided ideal of R such that every element of $N(R)$ is nilpotent.

Recall that the prime radical of M is the intersection of all prime submodules of M and is denoted by $P(M)$. The prime radical of a ring R is the intersection of all prime ideals of R and is denoted by $P(R)$.

Theorem 2.5 ([16]): Let I_1 and I_2 be two ideals of a ring R and let $I_1 + I_2 = \{a_1 + a_2 : a_1 \in I_1, a_2 \in I_2\}$. Then $I_1 + I_2$ is an ideal of R.

For convenience, we propose a theorem of nil right ideals over associative arbitrary rings here.

Theorem 2.6: If R is a ring and I, J are two nil right ideals of R, then the sum $(I + J)$ is a nil right ideal.

Proof. Let $I = \{a_1, a_2, a_3, \cdots, a_s\}$ and $J = \{b_1, b_2, b_3, \cdots, b_t\}$ be such that $a_1^{n_1} = 0, a_2^{n_2} = 0, a_1^{n_3} = 0, \cdots, a_s^{n_s} = 0$ where $n_1 \ge n_2 \ge n_3 \ge \cdots \ge n_s$ and $b_1^{m} = 0, b_2^{m_2} = 0, b_1^{m_3} = 0, \cdots, b_t^{m_t} = 0$ where $m_1 \ge m_2 \ge m_3 \ge \cdots \ge m_t$.

Let n and m be positive numbers such that $n = n_1 \ge n_i, \forall i$ and $m = m_1 \ge m_j, \forall j$, hence $a_i^n = 0, \forall i$ and $b_j^m = 0, \forall j$.

Since $\forall a_i \in I$ we have $a_i r \in I$ implies $a_i r = a_k$ where $a_k^k = a_k^n = 0; n \ge k$.

Also, $\forall b_j \in J$ we have $b_j r \in J$ implies $b_j r = b_t$ where $a_t^t = a_t^m = 0; m \ge t$.

Take for example $n = 3, m = 2$.

Also let $a \in I$ such that $a^3 = 0$ and $b \in J$ such that $b^2 = 0$.

So, as $n = 3$, we get $(ab)^3 = (a^2 b)^3 = (aba)^3 = \cdots = 0$, where $ab, a^2 b, aba \in I$.

Similarly, as $m = 2$, we get $(ba)^2 = (ba^2)^2 = (b^2 a)^2 = \cdots = 0$, where $ba, ba^2, b^2 a \in J$.

Now $I + J = \{a + b : a \in I, b \in J\}$. Then

$$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ba$$

$$\begin{aligned}(a+b)^3 &= a^3 + a^2 b + aba + ab^2 + ba^2 + bab + b^2 a \\ &= a^2 b + aba + ab^2 + ba^2 + bab\end{aligned}$$

$$(a+b)^4 = a^2 ba + aba^2 + abab + ba^2 b$$

$$(a+b)^5 = a^2 ba^2 + a^2 bab + aba^2 b + ba^2 ba$$

$$(a+b)^6 = a^2 ba^2 b + aba^2 ba + aba^2 b + ba^2 ba$$

$$(a+b)^7 = a^2 ba^2 ba$$

$$(a+b)^8 = ba^2 ba^2 ba = (ba^2) ba = 0$$

If we take $n = 3$ and $m = 3$, then we get

$$(a+b)^{19} = 0.$$

So if $a_i^n = 0; i = 1, 2, 3, \cdots, s$ and $b_j^m = 0; j = 1, 2, 3, \cdots, t$

Then there exists $n \ge n_i\ \forall i$ and $m \ge m_j\ \forall j$ such that

$$a_i^n = (a_i r)^n = 0, \forall i; a_i r \in I, b_j^m = (b_j r)^m = 0, \forall j; b_j r \in J$$

Then for any $a \in I, b \in J$ there exists k such that $(a+b)^k = 0$.

Thus the theorem is proved.

3. Nil and Nilpotent Modules

We see that the vector spaces are just special types of modules which arise when the underlying ring is a field. If R is a ring, the definition of an R-module M is closely analogous to the definition of a group action where R plays the role of the group and M the role of the set. The additional axioms for a module require that M itself have more structure (namely that M is an abelian group). Modules are the "representation objects" for rings, that is, they are, by definition, algebraic objects on which rings act. As the theory develops, it will become apparent how the structure of the ring R is reflected by the structure of its modules and vice versa.

In [11], many basic properties of nil and nilpotent modules and submodules have been given. In this paper, we give some more properties of nil and nilpotent modules and submodules in some special cases.

We first begin with the proposition that shows some properties of nil and nilpotent modules and submodules similar to that of nil and nilpotent rings.

The following theorem is an extension of the above theorem for modules over associative endomorphism rings.

Theorem 3.2: Let R be a ring with identity and with DCC on right ideals. Let N be the radical of R and let M be an R-module. Then $MN = 0$ if and only if M is the sum of irreducible submodules.

Proof. Let M is the sum of irreducible submodules, then any $m \in M$ is in $\sum_{k=1}^{n} M_k$, where M_k are irreducible.

Now $M_k N = M_k$ or, $M_k N = 0$. If $N^j = 0$, then $M_k N = M_k$ implies $M_k = 0$, a contradiction.

Thus $M_k N = 0$, where $mN = 0$ and so $MN = 0$.

Conversely, suppose that $MN = 0$. Then we can consider M as R/N-module, by putting

$m(r+N) = mr$ for all $r \in R$. Now R/N is semisimple and so, M is the sum of irreducible R/N-modules. Now let $\bar{M}$ be an irreducible R/N module, then since $\bar{M}N = 0$, $\bar{M}$ is an R-module, where $mr = m(r+N)$. Moreover, $\bar{M}$ has no non zero proper R-submodules, since this would induce proper non zero R/N-submodules. Thus $\bar{M}$ is an irreducible R-module.

Theorem 3.1 ([18]): Let M be an R-module, where R is a semisimple. Then M is the sum of irreducible submodules.

The following propositions and theorems give some properties of nil and nilpotent modules.

Proposition 3.3: Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a simple submodule of M. Then either $I_X^2 = 0$ or $X = f(M)$ for some idempotent $f \in I_X$.

Proof. Since X is a simple submodule of M, I_X is a minimal right ideal of S. Suppose that $I_X^2 \neq 0$. Then there is a $g \in I_X$ such that $gI_X \neq 0$. Since gI_X is a right ideal of S and $gI_X \subset I_X$, we have $gI_X = I_X$ by the minimality of I_X. Hence there exists $f \in I_X$ such that $gf = g$. The set $I = \{h \in I_X : gh = 0\}$ is a

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right ideal of S and I is properly contained in I_X since $f \notin I$. By the minimality of I_X, we must have $I = 0$. It follows that $f^2 - f \in I_X$ and $g\left(f^2 - f\right) = 0$, and hence $f^2 = f$. Note that $f(M) \subset X$ and $f(M) \neq 0$, and from this we have $f(M) = X$.

Proposition 3.4: Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M satisfies the ACC on fully invariant submodules, then $P(M)$ is nilpotent.

Proof. If M satisfies the ACC on fully invariant submodules, then S satisfies the ACC on two-sided ideals. Indeed, $I_1 \subset I_2 \subset \cdots$ is an ascending chain of two-sided ideals of S, then $I_1(M) \subset I_2(M) \subset \cdots$ is an ascending chain of fully invariant submodules of M. Since M has the ACC on fully invariant submodules, there exists a positive integer n such that $I_n(M) = I_k(M)$ for all $k > n$. Thus $I_n = I_k$ for all $k > n$, showing that S satisfies the ACC on two-sided ideals. Therefore $P(S)$ is nilpotent. Since $(S) = I_{P(M)}$, we have $P(M)$ is nilpotent.

Theorem 3.5: Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then M is a semiprime module if and only if M contains no nonzero nilpotent submodules.

Proof. By hypothesis, 0 is a semiprime submodule of M. If X is a nilpotent submodule of M, then $I_X^n = 0$ for some positive integer n, and hence $I_X^n(M) = 0$.

Note that $I_X(M) = 0$, we can see that $X = 0$.

Conversely, suppose that M contains no nonzero nilpotent submodules. Let I be an ideal of S such that $I^2(M) = 0$. Then we can write $I = I_{I(M)}$ and hence $I_{I(M)}^2 = 0$. It follows that $I(M)$ is a nilpotent submodule of M and we get $I(M) = 0$. Thus 0 is a semiprime submodule of M and thus M is a semiprime module.

Theorem 3.6: Let M be a quasi-projective, finitely generated right R-module which is a self-generator and $P(M)$ be the prime radical of M. If M is a noetherian module, then $P(M)$ is the largest nilpotent submodule of M.

Proof. Let $\mathcal{F}$ be the family of all minimal submodules of M. Then we can we write $P(M) = \bigcap_{X \in \mathcal{F}} X$. But $P(M)$ contains all nilpotent submodules of M. Again $I_{P(M)} = \bigcap_{X \in \mathcal{F}} I_X = P(S)$. Note that from our assumption we can see that S is a right noetherian ring. Then there exist only finitely many minimal prime ideals of S and there is a finite product of them which is 0, say $P_1 \cdots P_n = 0$. Since $I_{P(M)}$ is contained in each $P_i, i = 1, \cdots, n$, we have $I_{P(M)}^n = 0$. Thus $P(M)$ is nilpotent.

4. Conclusion

Nil and nilpotent rings and modules are very essential part of Abstract algebra. In the class of noetherian ring, nil ideals are nilpotent. Many properties of nil and nilpotent ideals of rings are not transferable to nil and nilpotent submodules. Modifying the structure of nil and nilpotent ideals we transferred the notions to modules. We also introduced a new concept of nil and nilpotent submodules.

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