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Approximation of Subsurface Seepage using 2- Dimensional Boussineq Equation

Chhaya K. Lande[1*]

[1]Symbiosis Institute of technology, Lavale, Pune, India.

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

Groundwater is the main source of fresh water available for human beings. The surface water groundwater interaction affects the quantity and quality of groundwater. Hence the study of surfacewater-groundwater interaction is the emerging topic in this new era. In this paper, the analytical approximation of water table fluctuation in the aquifer is presented. The aquifer is subjected to the recharge and withdrawal activity through multiple basins and wells in the domain. The time dependent multiple recharge is considered. The flow is approximated by a non linear partial differential equation called Boussineq equation. The solution of Boussineq equation is developed using Finite Fourier cosine transform. Response of the solution to using numerical examples has been tested. Effect of aquifer parameters on the fluctuation of water table formation mainly water mound and cone of depression due to recharge and withdrawal are presented. The effect of permeability of aquifer base on the water table is also discussed.

Keywords: Leaky base; Boussinesq equation; recharges; withdrawal; Fourier Transform.

**Corresponding author: Email: chhaya.lande@sitpune.edu.in;*

1 Introduction

Study of water pumping from multiwells and aquifer recharge systems plays an important role in management of drainage systems. Operations in multi wells and recharge affected by nearby water bodies too. For sustainable development of these systems, It is essential to have proper knowledge of the geological system, aquifer response to withdrawal and recharge activity for water management. To get prior knowledge of changes in groundwater and distribution of water in aquifers is difficult. In such complex geological systems mathematical models are one of the best tools which helps groundwater managers to understand the phenomenon better.

Mathematical models have ability to predict the transient as well as steady-state behavior of water table in unconfined aquifers under seepage and recharge conditions. These models might represent a simplified version of a hydrogeological system; reasonable alternative scenarios can be predicted, tested, and compared with their results. The first step in this direction is taken by Theis (1941), his study on the interaction of river and aquifer inspires many researchers to worked on this field. Mathematical models using several techniques for simulation of subsurface seepage flow in confined as well as unconfined aquifers have been presented by many investigators Polubarinova-Kochina [1], Hantush [2], Marino [3], Lockington [4], Hunt [5] etc. Most of these studies use a nonlinear Boussinesq equation as the central tool for estimation of surface-groundwater interaction. Alternative approximation based on kinematic wave theory was used by Baven [6], and Troch et al. (2002); Use of Boussinesq equation for approximation of subsurface seepage flow is still a preferred choice of researchers for its simple hydrologic approach [7,8], (Verhoest and Troch 2000; Upadhyaya and Chauhan 2001; Verhoest et al. 2002; Rai and Manglik (2012); Bansal [9,10,11].

Some fundamental works concerning water table fluctuations in a rectangular shaped homogeneous aquifer system due to localized recharge and withdrawal include Hantush [12], Hunt [13], Latinopoulos [14], Latinopoulos [15], Finnemore [16], Manglik et al. [17]. These studies use constant or time-varying recharge rate to simulate 'field like conditions'. Zomorodi (1991) presented a numerical study for evaluation of groundwater mound by including the effects of unsaturated zone on the infiltration rate and the role of in-transit water in reducing the fillable pore space. The analysis presented by him revealed that if rate of recharge is considered as constant, the predicted values might seriously underestimate or overestimate the actual results. Some researchers (Rai and Manglik 1999, 2012) propose a method consisting of sequence of line segments to approximate the rate of recharge.

Most of the existing models are based on a restrictive assumption that the aquifer is underlain by a perfectly impervious base. Aquifers in deep sedimentary basins are often multilayered with leaky confining beds. Hydraulic connection between two layers mediated by the confining semi pervious layer plays a critical role in determination of regional groundwater budget. When water is pumped out from such aquifers, the drawdown is partially supplemented by the underlying aquitard (Zlotnik and Tartakovsky, 2008). Recently, Bansal and Teloglou [18] analyzed the variations in water table in an unconfined aquifer overlying a semi pervious base due to multiple recharge and withdrawal. They considered an L-shaped aquifer which is homogeneous and isotropic. The analysis presented by them indicates significant variations in groundwater mound and cone of depressions on account of vertical leakage through the leaky base. However, the assumption of L-shaped boundary and isotropy narrows the applicability of their results. Tang [19] proposed a general approximate method to predict the aquifer response subject to water level variations in a free water body. Shaikh et al. [20] analyzed the dynamic behavior of tide induced water table variations in an unconfined aquifer system. Similarly Lande and Bansal [21,22] developed a model which predicts the transient behavior of water table, under recharge and withdrawal conditions where vertical recharge is approximated by exponential decaying function. The analysis is done by considering 2-Dimensional rectangular shape aquifers with leaky base. Finite Fourier transform is used to solve partial differential equations and variation in water mound as well as cone of depression is observed.

The present study is devoted to the new analytical solution of 2-dimensional linearized Boussinesq equation developed under the archetypical hydrological situation. The hydrological setting of the model consists of an unconfined anisotropic isolated aquifer overlaying a semipervious (leaky) base, subjected to recharge and withdrawal activities through multiple recharge basins and extraction/injection wells. The Parabolic partial differential equation is solved using finite Fourier transform to obtain the closed form expressions for water head distribution in the aquifer. Finally, the hypothetical examples illustrated the applicability and validity of the new result. Sensitivity of the hydraulic head based on variation in aquifer parameters is also analyzed.

2 Development of Mathematical Model

Schematic diagram of the problem is given in Fig. 1. An anisotropic unconfined aquifer of dimension *A* x *B* is underlain by a semipervious (leaky) base.

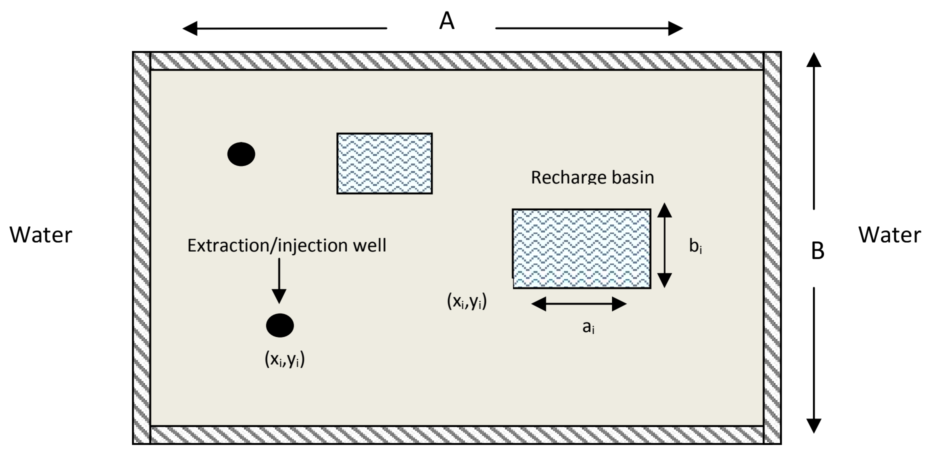

Fig. 1. Overview of a two-dimensional anisotropic isolated aquifer with multiple recharge basins, injection and extraction wells

Typically, such leaky semi-porous formations connect the unconfined aquifer with adjacent confined aquifers. The water table in the aquifers often affected by the water percolation from the base in the downward direction. The aquifer is anisotropic hence the hydraulic conductivities of the aquifer along *x* and *y* directions are K_x and K_y respectively.

$$K_x \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+K_y \frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)+P(x,y,t)=S\frac{\partial h}{\partial t}+\frac{k}{b}(h-h_0) \tag{1}$$

where S denotes the specific yield of the aquifer. *k* is the hydraulic conductivity of the leaky base and *b* denote the thickness of the base. The aquifer is subjected to the activities like time dependent recharge in the recharge basins and withdrawal from the wells. The term $P(x, y, t)$ signifies combined effects of recharge and withdrawal due to replenishment/extraction from arbitrary located rectangular basins and pumping wells. In the present work, the number of basins and wells are considered to be p_1 and p_2 respectively. We assume that the i^{th} basin is centered at (x_i, y_i) and is of dimension a_i x b_i; whereas the j^{th} well is located at (x_j, y_j). Dimension of wells are much small compared to that of basins. Recharge is considered at time-varying rate, whereas the extraction/injection is at constant rate. Thus, we define

$$P\left(x,\ y,\ t\right)=\left[\sum_{i=1}^{p_1} R_i\left(x,\ y,\ t\right)+\sum_{j=1}^{p_2}\omega_j\, Q_j\, \delta\left(x-x_j\right)\delta\left(y-y_j\right)\right] \tag{2}$$

Where $R_i(x, y, t)$ denotes the transient recharge rate in the i^{th} basin ($i = 1, 2, \ldots, p_1$) extending from $x_i \le x \le x_i + a_i$; $y_i \le y \le y_i + b_i$. The term ω_j is 1 or –1 according as the j^{th} well corresponds to a injection or extraction well. Q_j is the rate of injection/extraction in the j^{th} well ($j = 1, 2, \ldots, p_2$). δ is the Dirac delta function. While the extraction rate is usually constant during a complete cycle of pumping; the rate of recharge typically depends on several hydrologic parameters. It is noteworthy that the recession limb of a recharge hydrograph bears significant resemblance with an exponentially decaying function of time. Thus, we assume that

$$R_i(x,y,t)=\begin{cases} N(t) & x_i \le x \le x_i + a_i\,;\; y_i \le y \le y_i + b_i \\ 0 & \text{otherwise} \end{cases} \tag{3 a}$$

Where N(t) -time dependent recharge is defined as

$$N(t)=\begin{cases} N_0 & 0<x<t_1 \\ N_1 & t_1<x<t_2 \\ N_2 & t_2<x<t_3 \\ N_3 & t_3>t \end{cases} \tag{3 b}$$

The initial and the boundary conditions are prescribed as follows:

$$h\left(x,y,t=0\right)=h_0 \tag{4}$$

$$\left(\frac{\partial h}{\partial x}\right)_{x=0}=0; \qquad \left(\frac{\partial h}{\partial x}\right)_{x=A}=0 \tag{5}$$

$$\left(\frac{\partial h}{\partial y}\right)_{y=0}=0; \qquad \left(\frac{\partial h}{\partial y}\right)_{y=B}=0 \tag{6}$$

Equation (1) is a second order partial differential equation of parabolic nature, often referred to as two-dimensional Boussinesq equation. Due to its nonlinearity, Boussinesq equation is analytically intractable. In order to find an approximate analytical solution of (1), we rewrite it in the form

$$K_x\frac{\partial^2 h^2}{\partial x^2}+K_y\frac{\partial^2 h^2}{\partial y^2}+2P\left(x,y,t\right)=S\left(\frac{1}{h}\frac{\partial h^2}{\partial t}\right)+\frac{2k}{b}\frac{\left(h^2-h_0^2\right)}{\left(h+h_0\right)} \tag{7}$$

Equation (7) is now linearized by replacing the term h associated with $\partial h^2/\partial t$ in the first bracket and the term $(h + h_0)/2$ of the right-hand side by the mean depth of the saturation $\hbar$. The value of $\hbar$ is obtained by successive application of the relation $\hbar = (h_0 + h_t)/2$ where h_0 is the initial water head and h_t is the water head at the current moment (Marino 1973). The initial approximation of $\hbar$ is taken as h_0. We obtain

$$\frac{\partial^2 h^2}{\partial x^2}+\frac{K_y}{K_x}\frac{\partial^2 h^2}{\partial y^2}+\frac{2}{K_x}P\left(x,\ y,\ t\right)=\frac{S}{K_x\,\hbar}\left(\frac{\partial h^2}{\partial t}\right)+\frac{k}{K_x\,b\,\hbar}\left(h^2-{h_0}^2\right) \tag{8}$$

Now, define $H(x, y, t) = h^2 - h_0^2$, we get

$$\frac{\partial^2 H}{\partial x^2}+\frac{K_y}{K_x}\frac{\partial^2 H}{\partial y^2}+\frac{2}{K_x}P\left(x,\ y,\ t\right)=\frac{S}{K_x\,\hbar}\frac{\partial H}{\partial t}+\frac{2k}{K_x\,b\,\hbar}H \tag{9}$$

The initial and boundary conditions read

$$H\left(x,\ y,\ 0\right)=0 \tag{10}$$

$$\left(\frac{\partial H}{\partial x}\right)_{x=0} = 0; \quad \left(\frac{\partial H}{\partial x}\right)_{x=A} = 0 \tag{11}$$

$$\left(\frac{\partial H}{\partial y}\right)_{y=0} = 0; \quad \left(\frac{\partial H}{\partial y}\right)_{y=B} = 0 \tag{12}$$

Equation (9) along with the boundary conditions (10) to (12) is solved using Fourier Cosine transform. Fourier cosine transform is defined as follows:

$$\xi(m, n, t) = F_{sc}\left\{H(x,y,t);(x, y) \rightarrow (m, n)\right\} = \int_{x=0}^{A}\int_{y=0}^{B} H(x, y, t)\cos\left(\frac{m\pi x}{A}\right)\cos\left(\frac{n\pi y}{B}\right)dy\,dx \tag{13}$$

The Finite Fourier cosine transform reduces equation (9) to the following form

$$-\beta_m{}^2\xi - \gamma_n{}^2\xi + \frac{2}{K_x}\overline{P}(m, n, t) = \frac{1}{\nu}\frac{d\xi}{dt} + c\,\xi \tag{14}$$

Where

$$\beta_m = \frac{m\pi}{A}, \gamma_n = \frac{n\pi}{B}\sqrt{\frac{K_y}{K_x}}, c = \frac{k}{K_x\, b\,\hbar} \text{ and } \nu = \frac{K_x\,\hbar}{S} \tag{15}$$

And

$$\overline{P}(m,n,t) = \left[\sum_{i=1}^{p_1}\Omega_i\, N(t) + \sum_{j=1}^{p_2}\omega_j\,\eta_j\,Q_j\right] \tag{16}$$

Where

$$\Omega_i = \frac{\sqrt{Kx}}{\beta_m\,\gamma_n\sqrt{Ky}}\left[\sin\left\{\beta_m\left(x_i + a_i\right)\right\} - \sin\left(\beta_m\,x_i\right)\right]\left[\sin\left\{\gamma_n\sqrt{\frac{Kx}{Ky}}\left(y_i + b_i\right)\right\} - \sin\left(\gamma_n\sqrt{\frac{Kx}{Ky}}\,y_i\right)\right] \tag{17}$$

and

$$\eta_j = \cos\left(\beta_m\,x_j\right)\cos\left(\gamma_n\sqrt{\frac{Kx}{Ky}}y_j\right) \tag{18}$$

equation (14) is solved using ordinary method, the solution obtained is

$$\xi(m,n,t) = \frac{2\nu}{K_x}\left[\sum_{i=1}^{p_1}\Omega_i\left\{\frac{N(t)}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)\right\} + \sum_{j=1}^{p_2}\frac{\omega_j\,\eta_j\,Q_j}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)\right] \tag{19}$$

The term used is $\alpha = \nu\left(\beta_m{}^2 + \gamma_n{}^2\right)$ and τ is the variable of integration.

$H(x, y, t)$ can now be obtained by using inverse Fourier cosine transform which is define as

$$H(m.n,t)=\frac{4}{AB}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\xi(m,n,t)\sin\left(\frac{m\pi x}{A}\right)\cos\left(\frac{n\pi y}{B}\right) \tag{20}$$

Thus, the solution of equation (8) reads

$$\begin{aligned} h^2 = h_0{}^2 + \frac{8\nu}{ABK_x}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\left(\beta_m x\right)\cos\left(\gamma_n\sqrt{\frac{K_y}{K_x}}y\right)\Bigg[\sum_{j=1}^{p_2}\frac{\omega_j\,\eta_j\,Q_j}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right) \\ +\sum_{i=1}^{p_1}\Omega_i\left\{\begin{array}{l}\dfrac{1}{\alpha+\nu c}\left(N_0\left(1-e^{-(\alpha+\nu c)t_1}\right)\right)+N_1\left(e^{-(\alpha+\nu c)t_2}-e^{-(\alpha+\nu c)t_1}\right) \\ +N_2\left(e^{-(\alpha+\nu c)t_3}-e^{-(\alpha+\nu c)t_2}\right)+N_3\left(-e^{-(\alpha+\nu c)t_3}\right)\end{array}\right\}\Bigg] \end{aligned} \tag{21}$$

3 Discussion of Results

To implement the results developed in this study, domain considered to have two recatagular basins namely B_1 and B_2. Each basin is of dimension of 10 m x 10 m, and centered at (35m, 35m) and (115m, 35m) respectively. Along with the basins two extraction wells are considered called W_1 and W_2 located at (35m, 75m) and (115m, 75m). both the basins are subjected to have vertical time varying recharge. The recharge rates considered here are N_o=2.5, N_1=3, N_2=3.5 and N_4 = 4 for consecutive four days(t_1=1,t_2=2,t_3=3and t_4=4).Similarly water is extracted from wells at a constant rates 40 and 30 m^3/day respectively. Average saturated depth of the aquifer is determined using an iterative relation $\hbar = (h_0 + h_t)/2$ where h_0 is the initial water head and h_t is the water head at the current moment (Marino 1973). Initial approximation of $\hbar$ is taken as h_0.

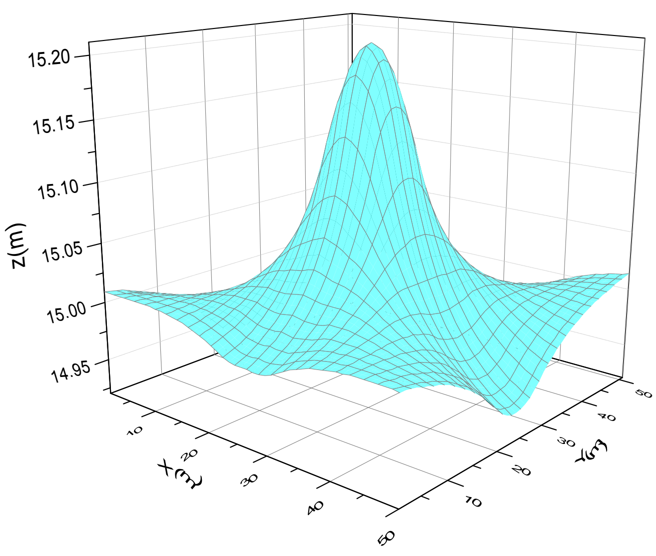

Fig. 2. a) Water mound for k=0.25

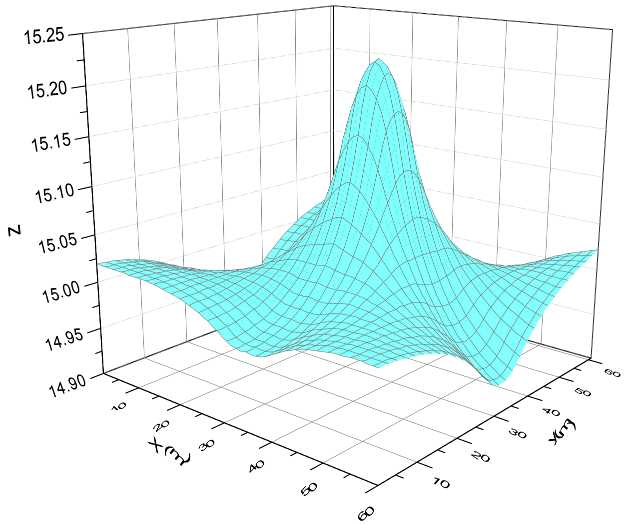

Fig. 2. b) Water mound for k=0

Transient profiles of the water table fluctuations are determined. Distribution of water head along line y = 35 m (line passing through the centers of recharge basins B_1 and B_2). It is observed that the groundwater mounds forms are symmetrical about the centers of the basins. Fig. 2 a & b shows the 3D view of the water mound form under recharge basin.

Fig. 2 a- shows the water mound form under recharge basin when the aquifer base is leaky(k=0.25) and in Fig. 2 b-is in case of impermeable base (k=0). In Fig. 2, The significant impact of hydraulic resistance of aquifer's base (measured by the ratio b/k) on the transient profiles of the phreatic surface is clearly seen. Numerical experiments reveal that the groundwater mound attains higher level in those aquifers which have comparatively higher values of hydraulic resistance. This is primarily due to vertical seepage loss through the aquifer's base, which decreases as the hydraulic resistance increases.

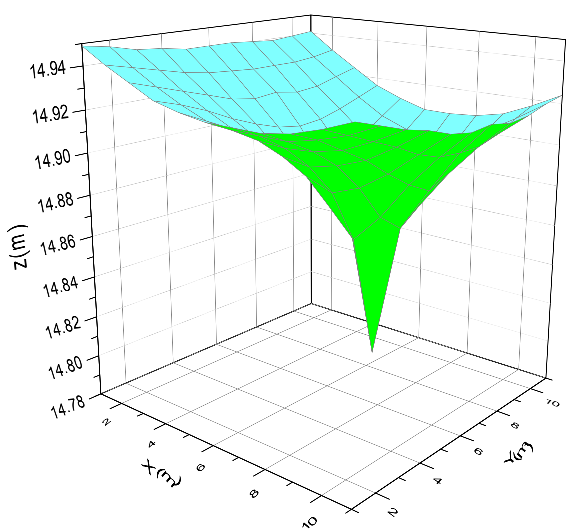

Fig. 3. a) Cone of depression when =0.25

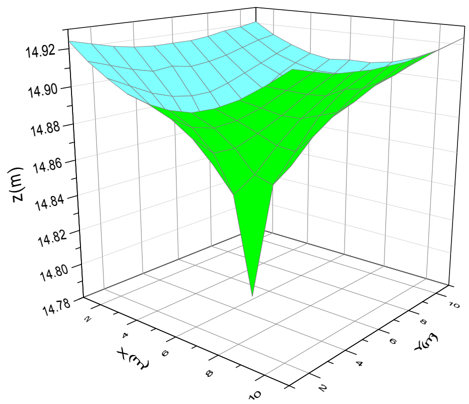

Fig. 3. b) cone of depression when k=0

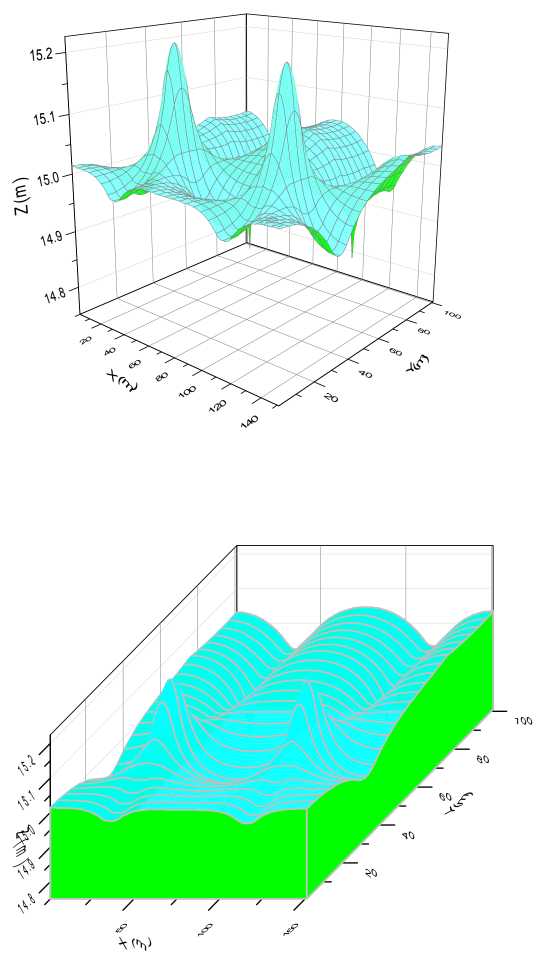

Fig. 4. a&b: 3D view of water profile for k=0.25 with cross sectional view

Lowering of water table due to continuous pumping from wells W_1 and W_2 are shown in Fig. 3 a & b . The difference in depth of cone under W_1 and W_2 is primarily due to varying pumping rate (Q_1 = 40 m^3/d, Q_2 = 30 m^3/d).

These profiles characterize cones of depression in the presence of semipervious base with k = 0.25 m/d. It can be observed from these figures that the depth of cone increases with time. Moreover, water table depletion induced by pumping from wells is also affected by the hydraulic resistance of the aquifer's base. When the base is leaky, withdrawal from the wells is supplemented by the leakage induced vertical flow from hydraulically connected sources. Consequently, the depth of the cone of depression is mitigated.

Three dimensional view of the groundwater mound and the cone of depression along with cross sectionl view is shown in Fig.. 4 a & b and Fig. 5 a & b for k = 0.25 m/d and fully impervious base (k=0) respectively. The changes in the water profile due to recharge and pumping is clearly seen in these fihures. The effect of hydraulic resistnce is also seen .

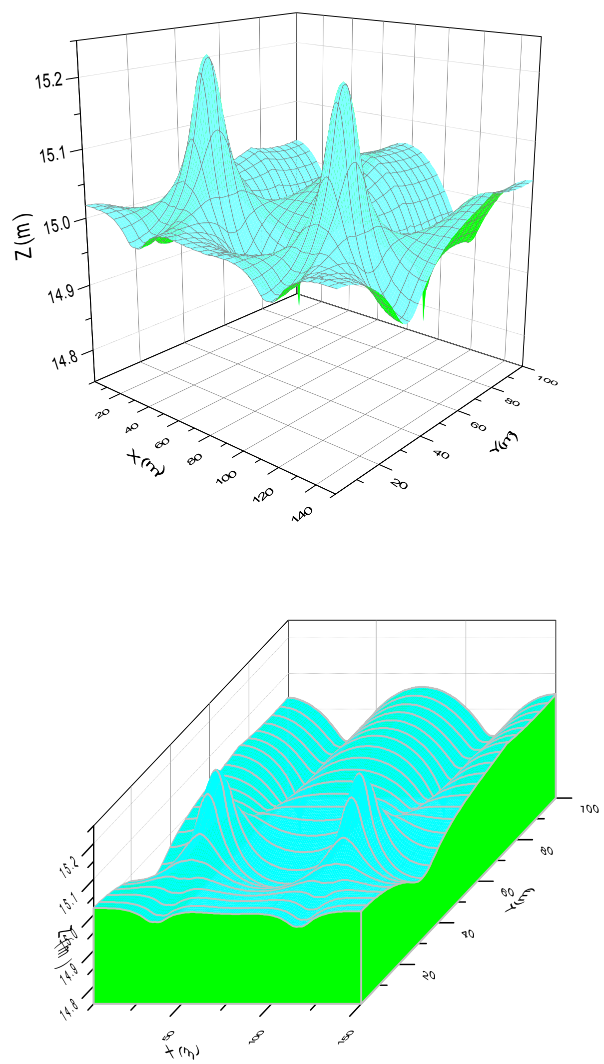

Fig. 5. a &b: 3D view of water profile for k=0 with cross sectional 3-D view.

4 Conclusion

Flow of subsurface seepage is governed by nonlinear Boussinesq equation which is analytically intractable. In this paper, approximate analytical solution of 2-dimensional Boussinesq equation is developed to simulate water table fluctuations in a rectangular shaped unconfined aquifer due to multiple recharge and withdrawal. The mathematical model consists of anisotropic and homogeneous aquifer system overlying a leaky base, and hydraulically connected with fully impervious from all four sides (Isolated). Analytical expressions for water head distribution are developed using finite Fourier transform. The solution developed in this study has the ability to predict the fluctuations in water table in unconfined aquifer due to multiple recharge and withdrawal. It is demonstrated with a numerical example that the semipervious layer supplements the drawdown beneath the wells, and reduces the height of groundwater mound beneath recharge basins.

The closed form solution obtained in this study solve the following purpose: (1) It can predict the water table variation induced by multiwells for pumping and vertical recharge, (2) It tests the aquifer sensitivity in response to different hydraulic parameters like recharge and pumping rate, base hydraulic conductivity etc.

Competing Interests

Author has declared that no competing interests exist.

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