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E-Bayesian and Hierarchical Bayesian Estimations Based on Dual Generalized Order Statistics from the Inverse Weibull Model

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Authors' contributions

This work was carried out in collaboration among all authors. Author HMR introduced the idea in a methodically structure, did the data analysis and drafted the manuscript. Author AMY assisted in building the study design and also did the final proofreading. Author SOA managed the analyses of the study and literature searches and also proofread the draft. All authors read and approved the final manuscript.

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Abstract

This paper is devoted to compare the E-Bayesian and hierarchical Bayesian estimations of the scale parameter corresponding to the inverse Weibull distribution based on dual generalized order statistics. The E-Bayesian and hierarchical Bayesian estimates are obtained under balanced squared error loss function (BSELF), precautionary loss function (PLF), entropy loss function (ELF) and Degroot loss function (DLF). The properties of the E-Bayesian and hierarchical Bayesian estimates are investigated. Comparisons among all estimates are performed in terms of absolute bias (ABias) and mean square error (MSE) via Monte Carlo simulation. Numerical computations showed that E-Bayesian estimates are more efficient than the hierarchical Bayesian estimates.

Keywords: E-Bayesian estimates; inverse Weibull distribution; hierarchical Bayesian estimates; loss functions; Monte Carlo simulation.

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1 Introduction

The inverse Weibull distribution has great importance in many applications including the dynamic components of diesel engines and several data sets such as the times to breakdown of an insulating fluid subject to the action of a constant tension (see Nelson [1]). Several authors have studied the inverse Weibull distribution; for examples, Calabria and Pulcini [2] have obtained two sample Bayesian prediction for inverse Weibull model in complete samples. Kundua and Howlader [3] have constructed one and two sample predictive posteriors of the future order statistics from inverse Weibull distribution based on type-ii censoring. Abd Ellah [4] has obtained the non-Bayesian and Bayesian estimates for parameters and reliability function for inverse Weibull distribution based on generalized order statistics. Singh et al. [5] have studied the classical and Bayesian approaches in estimating the unkown parameters of inverse Weibull distribution under type-i and type-ii censored data. Singh et al. [6] have discussed the Bayesian technique for prediction of the future samples from inverse Weibull distribution based on type-ii hybrid censoring. Yahgmaei et al. [7] have used the maximum likelihood and Bayesian schemes to estimate scale parameter of the inverse Weibull distribution. The probability density function (pdf) and cumulative distribution function (cdf) of the inverse Weibull distribution (IWD) are given respectively as

$$f(x) = \lambda \theta x^{-\lambda-1} e^{-\theta x^{-\lambda}}, \qquad x > 0,\ \lambda, \theta > 0, \tag{1}$$

$$F(x) = e^{-\theta x^{-\lambda}}, \qquad x > 0,\ \lambda, \theta > 0. \tag{2}$$

where θ and λ are the scale and shape parameters respectively.

Order statistics are widely used in statistical inferences and modelling purposes. Kamps [8] introduced the concept of the generalized order statistics (**GOS**) as a unified scheme to many models of ordered random variables such as ordinary order statistics, upper record values and sequential order statistics. Burkschat et al. [9] proposed the dual generalized order statistics (**DGOS**) as a dual model of **GOS** and a unification of various models of decreasingly ordered random variables such as reversed order statistics, lower record values and lower pfeifer record. Given a random sample drawn from an absolutely continuous distribution function (cdf) F with corresponding probability density function (pdf) f , the joint density function of the first n **DGOS** can be written as

$$f^{X(1,n,m,k),\ldots X(n,n,m,k)}(x_1,\ldots,x_n) = C_{n-1}\left(\prod_{i=1}^{n-1}(F(x_i))^m f(x_i)\right)(F(x_n))^{k-1} f(x_n) \tag{3}$$

On the cone $F^{-1}(0) < x_n \le \ldots \le x_1 < F^{-1}(1)$ with parameters $n \in N, n \ge 2, k > 0, m \ge -1$ and $C_{n-1} = \prod_{i=1}^{n} \gamma_i$ such that $\gamma_i = k + (n-i)(m+1) > 0$ for all $i \in \{1,\ldots,n\}$. The lower record values of size n can be derived from the **DGOS** procedure as a special case by taking $m = -1$ and $k = 1$.

The E-Bayesian is a new approach of estimation was first proposed by Han [10]. Many authors studied the E-Bayesian technique such as, Han [11] estimated the reliability parameter of the exponential model by using the E-Bayesian and hierarchical Bayesian methods under type-i censored samples and by considering the quadratic loss function.Yin and Liu [12] used the E-Bayesian and hierarchical Bayesian estimation schemes in estimating the reliability parameter of geometric distribution under scaled squared loss function in complete samples. Wei et al. [13] obtained the E-Bayesian and minimum risk equivariant estimates for the Burr-XII distribution based on entropy loss function in complete samples. Jaheen and Okasha [14] estimated the parameter and reliability function of Burr-XII distribution by using the E-Bayesian and Bayesian methods under squared error and LINEX loss functions based on type-ii censored data. Cai et al. [15] constructed the E-Bayesian approach for forecasting of security investment. Okasha [16] derived the maximum likelihood, Bayesian and E-Bayesian estimators for the parameter, reliability and hazard functions associated to the Weibull distribution under type-ii censoring. Azimi et al. [17] obtained the E-Bayesian and Bayesian estimates for parameter and reliability function of the generalized half Logistic distribution under progressively type-ii censored data and by using symmetric and asymmetric loss functions. Javadkani et al. [18] used the Bayesian, empirical Bayesian and E-Bayesian methods in estimating shape parameter and reliability function of the two parameter bathtub-shaped lifetime distribution based on progressively first-failure-censoring and by using the minimum expected and LINEX loss functions. Liu et al. [19] derived the E-Bayesian and hierarchical Bayesian estimates for the Rayleigh distribution under q-symmetric entropy loss function in complete samples. Reyad and Othman [20] obtained the Bayesian and E-Bayesian estimates for shape parameter of the Gumbell type-ii model based on type-ii censoring and by considering squared error, LINEX, Degroot, quadratic and minimum expected loss functions. Reyad and Othman [21] derived the E-Bayesian and Bayesianestimators for the Kumaraswamy distribution under type-ii censored data and by using different symmetric and asymmetric loss functions. Reyad et al. [22] compared the E-Bayesian, hierarchical Bayesian, Bayesian and empirical Bayesian estimates of shape parameter and hazard function cooresponding to the Gompertz model under type-ii censoring and by using squared error, quadratic, entropy and LINEX loss functions. Reyad et al. [23] obtained the QE-Bayesian, quasi-bayesian, quasi-hierarchical Bayesian and quasi-empirical Bayesian estimates for the scale parameter of the Erlang distribution under different loss functions in complete samples. Reyad et al. [24] compared the QE-Bayesian and E-Bayesian estimation methods in estimating the scale paramter of the Frechet distribution based on squared error, entropy, weighted balanced and minimum expected loss functions in complete samples.

The paper aims to compare the E-Bayesian and hierarchical Bayesian approaches in estimating scale parameter associated to the inverse Weibullmodel based on **DGOS**. The resulting estimates are derived under different loss functions and specialized to lower record values. The properties of the E-Bayesian estimates are studied and the relations among E-Bayesian and hierarchical Bayesian estimates are investigated.

The remainder of this study is organized as follow. In Section 2, the various loss functions will be used in this study are viewed and posterior distribution is derived. The E-Bayesian estimates for θ are derived under BSELF, PLF, ELF and DLF in Section 3. In Section 4, the hierarchical Bayesian estimates for θ are obtained under BSELF, PLF, ELF and DLF. In Section 5, the properties of the E-Bayesian and hierarchical Bayesian estimates are investigated. In Section 6, numerical computations are used to assess the performance of the resulting estimates. Finally, some concluding remarks are presented in Section 7.

2 The Loss Functions and Posterior Distribution

In this section, we referred to the various loss functions concerned in this paper and derived the posterior distribution associated to IWD.

2.1 The loss functions

We will use the following loss functions:

2.1.1 The balanced squared error loss function (BSELF)

Ahmadi et al. [25] defined the balanced sequred error loss function (BSELF)to be:

$$L_1(\theta,\hat{\theta}) = w\,(\hat{\theta}-\theta')^2 + (1-w)(\hat{\theta}-\theta)^2 . \tag{4}$$

where w is a suitable positive weight function, $\hat{\theta}$ is an estimator of θ, θ' is a prior estimator of θ obtained usually by either maximum likelihood or least squares methods and $(\hat{\theta}-\theta)^2$ is the squared error loss function. The Bayes estimator of θ relative to the BSELF denoted by $\hat{\beta}_{BB}$ can be obtained as

$$\hat{\theta}_{BB} = w\,\theta' + (1-w)E_h(\theta|\underline{x}) . \tag{5}$$

provided that the expectation corresponding to posterior distribution; $E_h(\theta|\underline{x})$ exisits and finite.

2.1.2 The precautionary loss function (PLF)

Nostrom [26] defined the precautionary loss function (PLF) as follows:

$$L_2(\theta,\hat{\theta}) = \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}} . \tag{6}$$

The Bayes estimator of θ based on PLF denoted by $\hat{\theta}_{BP}$ can be obtained as

$$\hat{\theta}_{BP} = \left[E_h(\theta^2|\underline{x})\right]^{\frac{1}{2}} . \tag{7}$$

provided that the expectation $E_h(\theta^2|\underline{x})$ exisits and finite.

2.1.3 The entropy loss function (ELF)

Day et al. [27] have discussed the entropy loss function (ELF) of the form:

$$L_3(\theta,\hat{\theta}) \propto \left(\frac{\hat{\theta}}{\theta}\right) - \ln\left(\frac{\hat{\theta}}{\theta}\right) - 1 . \tag{8}$$

where $\hat{\theta}$ is an estimator of θ. The Bayes estimator of θ relative to ELF denoted by $\hat{\theta}_{BE}$ can be obtained as

$$\hat{\theta}_{BE} = \left[E_h(\theta^{-1}|\underline{x})\right]^{-1} . \tag{9}$$

provided that the expectation $E_h(\theta^{-1}|\underline{x})$ exisits and finite.

2.1.4 The degroot loss function (DLF)

The Degroot loss function (DLF) is introduced by Degroot [28] to be:

$$L_4(\theta,\hat{\theta}) = \left(\frac{\theta-\hat{\theta}}{\hat{\theta}}\right) . \tag{10}$$

The Bayes estimator relative to DLF denoted by $\hat{\theta}_{BD}$ can be obtained as

$$\hat{\theta}_{BD} = \frac{E_h(\theta^2|\underline{x})}{E_h(\theta|\underline{x})} . \tag{11}$$

provided that the expectations $E_h(\theta^2|\underline{x})$, $E_h(\theta|\underline{x})$ are exisit and finite.

2.2 The posterior distribution

Assume that $X(1,n,m,k),...,X(n,n,m,k)$ be n **DGOS** taken from IWD, the likelihood function can be obtained by subsituting from Eqs. (1) and (2) in Eq. (3) to be

$$L(\lambda,\theta|\underline{x}) \propto \lambda^n \theta^n \left(\prod_{i=1}^{n} x_i^{-\lambda-1}\right) \exp\left[(-\theta)\left((m+1)\sum_{i=1}^{n-1} x_i^{-\lambda} + kx_n^{-\lambda}\right)\right]. \tag{12}$$

Assuming λ is known, then the likelihood function in Eq. (12) become

$$L(\theta|\underline{x}) \propto \theta^n e^{-\theta H} . \tag{13}$$

Where

$$H = (m+1)\sum_{i=1}^{n-1} x_i^{-\lambda} + kx_n^{-\lambda} . \tag{14}$$

We can use the exponential distribution as a conjugate prior distribution of θ with rate parameter a and its pdf given by

$$g(\theta|a) = a\, e^{-a\theta}, \qquad \theta > 0,\ a > 0 \tag{15}$$

From the Bayes theorem**,** the posterior distribution of θ can be obtained by combining Eqs. (13) and (15) to be

$$h(\theta|\underline{x}) = \frac{L(\theta|\underline{x})\, g(\theta|a)}{\int_0^\infty L(\theta|\underline{x})\, g(\theta|a)\, d\theta} = \frac{(H+a)^{n+1}}{\Gamma(n+1)} \theta^n e^{-\theta(H+a)}, \qquad \theta, a > 0 . \tag{16}$$

That mean, the posterior distribution of θ obeys $\Gamma(H+a, n+1)$.

3 The E-Bayesian estimation

In this section, we have obtained the E-Bayesian estimates for scale parameter of IWD under BSEL, PLF, ELF and DLF.

Based on Han [29], the hyperparameter a must be selected to guarantee that $g(\theta|a)$ given in Eq. (15) is a decreasing function of θ. The derivative of $g(\theta|a)$ with respect to θ is

$$\frac{dg(\theta|a)}{d\theta} = (-a^2)\exp[-a\theta] . \tag{17}$$

Note that $a > 0$ and $\theta > 0$ leads to for any value of $0 < a < \infty$, imply to $\frac{dg(\theta|a)}{d\theta} < 0$, and therefore $g(\theta|a)$ is a decreasing function of θ. Consequently, it is convention to choose the hyperparameter a under the restriction $0 < a < c$, where c is a given upper bound (c is a positive constant). Then, we can use the following hyperprior distributions of a introduced by Han [30].

$$\pi_1(a) = \frac{2(c-a)}{c^2}, \qquad 0 < a < c, \tag{18}$$

$$\pi_2(a) = \frac{1}{c} \qquad 0 < a < c. \tag{19}$$

And

$$\pi_3(a) = \frac{2a}{c^2}, \qquad 0 < a < c. \tag{20}$$

3.1 The E-Bayesian estimation under BSELF

Theorem 1. Assuming BSELF in Eq. (4), the posterior distribution in Eq. (16) and the hyperprior distributions of a in Eqs. (18), (19) and (20), we have two conclusions:

(i) The Bayesian estimate $\hat{\theta}_{BB}$ of θ based on BSELF is

$$\hat{\theta}_{BB} = \frac{wn}{H} + (1-w)\left(\frac{n+1}{H+a}\right). \tag{21}$$

(ii) The E-Bayesian estimates $\hat{\theta}_{EBB1}$, $\hat{\theta}_{EBB2}$ and $\hat{\theta}_{EBB3}$ of θ based on $\pi_1(a)$, $\pi_2(a)$ and $\pi_3(a)$ respectively relative to BSELF are the following:

$$\hat{\theta}_{EBB1} = \frac{wn}{H} + \left[\frac{2(1-w)(n+1)}{c}\right]\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right], \tag{22}$$

$$\hat{\theta}_{EBB2} = \frac{wn}{H} + \left[\frac{(1-w)(n+1)}{c}\right]\ln\left(1+\frac{c}{H}\right). \tag{23}$$

And

$$\hat{\theta}_{EBB3} = \frac{wn}{H} + \left[\frac{2(1-w)(n+1)}{c}\right]\left[1-\left(\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)\right]. \tag{24}$$

Proof. (i) The natural logarithm of the likelihood function in Eq. (13) is given by

$$\ell(\theta) \propto n\ln\theta - \theta H. \tag{25}$$

Differentiating Eq. (25) with respect to θ and equating the result to zero, then the likelihood equation for θ is given by

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{n}{\theta} - H = 0. \tag{26}$$

The maximum likelihood estimator θ' of θ can be obtained by solving Eq. (26) to be

$$\theta' = \frac{n}{H}. \tag{27}$$

We can derive $E_h(\theta|\underline{x})$ by using Eq. (16) to be

$$E_h(\theta|\underline{x}) = \int_0^\infty \theta \left[\frac{(H+a)^{n+1}}{\Gamma(n+1)} \theta^n e^{-\theta(H+a)} \right] d\theta = \frac{n+1}{H+a}. \tag{28}$$

Consequentely, the Bayesian estimate $\hat{\theta}_{BB}$ can be obtained by using Eqs. (27) and (28) in Eq. (5) to be

$$\hat{\theta}_{BB} = \frac{wn}{H} + (1-w)\left(\frac{n+1}{H+a}\right).$$

(ii) The E-Bayesian estimate $\hat{\theta}_{EBB1}$ based on $\pi_1(a)$ can be obtained by using Eqs. (18) and (21) to be

$$\hat{\theta}_{EBB1} = \int_0^c \left[\frac{wn}{H} + (1-w)\left(\frac{n+1}{H+a}\right)\right] \frac{2(c-a)}{c^2} da = \frac{wn}{H} + \left[\frac{2(1-w)(n+1)}{c}\right]\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right].$$

Similarly, the E-Bayesian estimates $\hat{\theta}_{EBB2}$, $\hat{\theta}_{EBB3}$ based on $\pi_2(a)$, $\pi_3(a)$ can be obtained by using Eqs. (19), (21) and Eqs. (20), (21) respectively to be

$$\hat{\theta}_{EBB2} = \int_0^c \left[\frac{wn}{H} + (1-w)\left(\frac{n+1}{H+a}\right)\right] \frac{1}{c} da = \frac{wn}{H} + \left[\frac{(1-w)(n+1)}{c}\right]\ln\left(1+\frac{c}{H}\right).$$

And

$$\hat{\theta}_{EBB3} = \int_0^c \left[\frac{wn}{H} + (1-w)\left(\frac{n+1}{H+a}\right)\right] \frac{2a}{c^2} da = \frac{wn}{H} + \left[\frac{2(1-w)(n+1)}{c}\right]\left[1-\left(\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)\right].$$

3.2 The E-Bayesian estimation under PLF

Theorem 2. Assuming PLF in Eq. (6), the posterior distribution in Eq. (16) and the hyperprior distributions of a in Eqs. (18), (19) and (20), we have two conclusion:

(i) The Bayesian estimate $\hat{\theta}_{BP}$ of θ based on PLF is

$$\hat{\theta}_{BP} = \frac{\sqrt{(n+1)(n+2)}}{H+a}. \tag{29}$$

(ii) The E-Bayesian estimates $\hat{\theta}_{EBP1}$, $\hat{\theta}_{EBP2}$ and $\hat{\theta}_{EBP3}$ of θ based on $\pi_1(a)$, $\pi_2(a)$ and $\pi_3(a)$ respectively relative to PLF are the following:

$$\hat{\theta}_{EBP1} = \frac{2\sqrt{(n+1)(n+2)}}{c}\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right], \tag{30}$$

$$\hat{\theta}_{EBP2} = \frac{\sqrt{(n+1)(n+2)}}{c} \ln\left(1+\frac{c}{H}\right). \tag{31}$$

And

$$\hat{\theta}_{EBP3} = \frac{2\sqrt{(n+1)(n+2)}}{c}\left[1-\left(\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)\right]. \tag{32}$$

Proof. (i) By using Eq. (16),we get

$$E_h\left(\theta^2 | \underline{x}\right) = \int_0^\infty \theta^2 \left[\frac{(H+a)^{n+1}}{\Gamma(n+1)}\theta^n e^{-\theta(H+a)}\right] d\theta = \frac{(n+1)(n+2)}{(H+a)^2}. \tag{33}$$

Then, the Bayesian estimate $\hat{\theta}_{BP}$ can be obtained by using Eq. (33) in Eq. (7) to be

$$\hat{\theta}_{BP} = \sqrt{\frac{(n+1)(n+2)}{(H+a)^2}} = \frac{\sqrt{(n+1)(n+2)}}{H+a}.$$

(ii) The E-Bayesian estimate $\hat{\theta}_{EBB1}$ based on $\pi_1(a)$ can be obtained by using Eqs. (18) and (29) to be

$$\hat{\theta}_{EBP1} = \int_0^c \left[\frac{\sqrt{(n+1)(n+2)}}{H+a}\right]\frac{2(c-a)}{c^2} da = \frac{2\sqrt{(n+1)(n+2)}}{c}\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right].$$

Similarly, the E-Bayesian estimates $\hat{\theta}_{EBB2}$, $\hat{\theta}_{EBB3}$ based on $\pi_2(a)$, $\pi_3(a)$ can be obtained by using Eqs. (19), (29) and Eqs. (20), (29) respectively to be

$$\hat{\theta}_{EBP2} = \int_0^c \left[\frac{\sqrt{(n+1)(n+2)}}{H+a}\right]\frac{1}{c} da = \frac{\sqrt{(n+1)(n+2)}}{c}\ln\left(1+\frac{c}{H}\right).$$

And

$$\hat{\theta}_{EBP3} = \int_0^c \left[\frac{\sqrt{(n+1)(n+2)}}{H+a}\right]\frac{2a}{c^2} da = \frac{2\sqrt{(n+1)(n+2)}}{c}\left[1-\left(\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)\right].$$

3.3 The E-Bayesian estimation under ELF

Theorem 3. Assuming ELF in Eq. (8), the posterior distribution in Eq. (16) and the hyperprior distributions of a in Eqs. (18), (19) and (20), we have two conclusion:

(i) The Bayesian estimate $\hat{\theta}_{BE}$ of θ based on ELF is

$$\hat{\theta}_{BE} = \frac{n}{H+a}. \tag{34}$$

(ii) The E-Bayesian estimates $\hat{\theta}_{EBE1}$, $\hat{\theta}_{EBE2}$ and $\hat{\theta}_{EBE3}$ of θ based on $\pi_1(a)$, $\pi_2(a)$ and $\pi_3(a)$ respectively relative to PLF are the following:

$$\hat{\theta}_{EBE1} = \frac{2n}{c}\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right], \tag{35}$$

$$\hat{\theta}_{EBE2} = \frac{n}{c}\ln\left(1+\frac{c}{H}\right). \tag{36}$$

And

$$\hat{\theta}_{EBE3} = \frac{2n}{c}\left[1-\left(\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)\right]. \tag{37}$$

Proof. (i) By usingEq. (16), we get

$$E_h\left(\theta^{-1}|\underline{x}\right) = \int_0^\infty \theta^{-1}\left[\frac{(H+a)^{n+1}}{\Gamma(n+1)}\theta^n e^{-\theta(H+a)}\right]d\theta = \frac{H+a}{n}. \tag{38}$$

Then, the Bayesian estimate $\hat{\theta}_{BE}$ can be obtained by using Eq. (38) in Eq. (9) to be

$$\hat{\theta}_{BE} = \left[\frac{H+a}{n}\right]^{-1} = \frac{n}{H+a}.$$

(ii) The E-Bayesian estimate $\hat{\theta}_{EBE1}$ based on $\pi_1(a)$ can be obtained by using Eqs. (18) and (34) to be

$$\hat{\theta}_{EBE1} = \int_0^c \left[\frac{n}{H+a}\right]\frac{2(c-a)}{c^2}da = \frac{2n}{c}\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right].$$

Similarly, the E-Bayesian estimates $\hat{\theta}_{EBB2}$, $\hat{\theta}_{EBB3}$ based on $\pi_2(a)$, $\pi_3(a)$ can be obtained by using Eqs. (19), (29) and Eqs. (20), (29) respectively to be

$$\hat{\theta}_{EBE2} = \int_0^c \left[\frac{n}{H+a}\right]\frac{1}{c}da = \frac{n}{c}\ln\left(1+\frac{c}{H}\right).$$

And

$$\hat{\theta}_{EBE3} = \int_0^c \left[\frac{n}{H+a}\right]\frac{2a}{c^2}da = \frac{2n}{c}\left[1-\left(\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)\right].$$

3.4 The E-Bayesian estimation under DLF

Theorem 4. Assuming DLF in Eq. (10), the posterior distribution in Eq. (16) and the hyperprior distributions of a in Eqs. (18), (19) and (20), we have two conclusion:

(i) The Bayesian estimate $\hat{\theta}_{BD}$ of θ based on PLF is

$$\hat{\theta}_{BD} = \frac{n+2}{H+a}. \tag{39}$$

(ii) The E-Bayesian estimates $\hat{\theta}_{EBD1}$, $\hat{\theta}_{EBD2}$ and $\hat{\theta}_{EBD3}$ of θ based on $\pi_1(a)$, $\pi_2(a)$ and $\pi_3(a)$ respectively relative to DLF are the following:

$$\hat{\theta}_{EBD1} = \frac{2(n+2)}{c}\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right], \tag{40}$$

$$\hat{\theta}_{EBD2} = \frac{(n+2)}{c}\ln\left(1+\frac{c}{H}\right). \tag{41}$$

And

$$\hat{\theta}_{EBD3} = \frac{2(n+2)}{c}\left[1-\left(\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)\right]. \tag{42}$$

Proof. (i) The Bayesian estimate $\hat{\theta}_{BD}$ can be obtained by using Eqs. (28) and (33) in Eq. (11) to be

$$\hat{\theta}_{BD} = \frac{(n+1)(n+2)/(H+a)^2}{(n+1)/(H+a)} = \frac{n+2}{H+a}.$$

(ii) The E-Bayesian estimate $\hat{\theta}_{EBB1}$ based on $\pi_1(a)$ can be obtained by using Eqs. (18) and (42) to be

$$\hat{\theta}_{EBD1} = \int_0^c \left[\frac{n+2}{H+a}\right]\frac{2(c-a)}{c^2}da = \frac{2(n+2)}{c}\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right].$$

Similarly, the E-Bayesian estimates $\hat{\theta}_{EBB2}$, $\hat{\theta}_{EBB3}$ based on $\pi_2(a)$, $\pi_3(a)$ can be obtained by using Eqs. (19), (29) and Eqs. (20), (29) respectively to be

$$\hat{\theta}_{EBD2} = \int_0^c \left[\frac{n+2}{H+a}\right]\frac{1}{c}da = \frac{(n+2)}{c}\ln\left(1+\frac{c}{H}\right)$$

And

$$\hat{\theta}_{EBD3} = \int_0^c \left[\frac{n+2}{H+a}\right]\frac{2a}{c^2}da = \frac{2(n+2)}{c}\left[1-\left(\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)\right].$$

4 The Hierarchical Bayesian Estimation

In this section, the hierarchical Bayesian estimates for scale parameter of IWD based on BSEL, PLF, ELF and DLF are derived.

Based to Lindley and Smith [31], if a is hyperparameter in θ, the prior density function of θ is $g(\theta|a)$ given in Eq. (15) and the hyperprior distributions of a are given in Eqs. (18), (19) and (20), then the corresponding hierarchical prior distributions of θ are given as the following:

$$\pi_4(\theta) = \int_0^c g(\theta|a)\pi_1(a)da = \frac{2}{c^2}\int_0^c a(c-a)e^{-a\theta}da, \tag{43}$$

$$\pi_5(\theta) = \int_0^c g(\theta|a)\,\pi_2(a)da = \frac{1}{c}\int_0^c a e^{-a\theta} da\,. \tag{44}$$

And

$$\pi_6(\theta) = \int_0^c g(\theta|a)\,\pi_3(a)da = \frac{2}{c^2}\int_0^c a^2 e^{-a\theta} da\,. \tag{45}$$

According to Bayesian theorem, the hierarchical posterior distributions of θ can be derived by combining Eqs. (13), (43), (44) and (45) to be

$$f_1(\theta|\underline{x}) = \frac{L(\theta|\underline{x})\,\pi_4(\theta)}{\int_0^\infty L(\theta|\underline{x})\,\pi_4(\theta)d\theta} = \frac{\int_0^c a(c-a)\theta^n\, e^{-\theta(H+a)}da}{\Gamma(n+1)\int_0^c a(c-a)(H+a)^{-(n+1)}da}, \tag{46}$$

$$f_2(\theta|\underline{x}) = \frac{L(\theta|\underline{x})\,\pi_5(\theta)}{\int_0^\infty L(\theta|\underline{x})\,\pi_5(\theta)d\theta} = \frac{\int_0^c a\theta^n\, e^{-\theta(H+a)}da}{\Gamma(n+1)\int_0^c a(H+a)^{-(n+1)}da}\,. \tag{47}$$

And

$$f_3(\theta|\underline{x}) = \frac{L(\theta|\underline{x})\,\pi_6(\theta)}{\int_0^\infty L(\theta|\underline{x})\,\pi_6(\theta)d\theta} = \frac{\int_0^c a^2\theta^n\, e^{-\theta(H+a)}da}{\Gamma(n+1)\int_0^c a^2(H+a)^{-(n+1)}da}\,. \tag{48}$$

4.1 The hierarchical Bayesian estimation under BSELF

Theorem 5. Assuming BSELF in Eq. (4), the hierarchical posterior distributions in Eqs. (46), (47) and (48), then the hierarchical Bayes estimates $\hat{\theta}_{HBB1}$, $\hat{\theta}_{HBB2}$ and $\hat{\theta}_{HBB3}$ of θ are the following:

$$\hat{\theta}_{HBB1} = \frac{wn}{H} + \frac{(1-w)(n+1)\int_0^c a(c-a)(H+a)^{-(n+2)}da}{\int_0^c a(c-a)(H+a)^{-(n+1)}da}, \tag{49}$$

$$\hat{\theta}_{HBB2} = \frac{wn}{H} + \frac{(1-w)(n+1)\int_0^c a(H+a)^{-(n+2)}da}{\int_0^c a(H+a)^{-(n+1)}da}\,. \tag{50}$$

And

$$\hat{\theta}_{HBB3} = \frac{wn}{H} + \frac{(1-w)(n+1)\int_0^c a^2(H+a)^{-(n+2)}da}{\int_0^c a^2(H+a)^{-(n+1)}da}\,. \tag{51}$$

Proof. We can derive $E_{f_i}(\theta|\underline{x}), (i=1,2,3)$ by using Eqs. (46), (47) and (48) to be

$$E_{f_1}(\theta|\underline{x}) = \frac{\int_0^\infty \theta\left[\int_0^c a(c-a)\theta^n e^{-\theta(H+a)} da\right] d\theta}{\Gamma(n+1)\int_0^c a(c-a)(H+a)^{-(n+1)} da} = \frac{(n+1)\int_0^c a(c-a)(H+a)^{-(n+2)} da}{\int_0^c a(c-a)(H+a)^{-(n+1)} da}, \tag{52}$$

$$E_{f_2}(\theta|\underline{x}) = \frac{\int_0^\infty \theta\left[\int_0^c a\theta^n e^{-\theta(H+a)} da\right] d\theta}{\Gamma(n+1)\int_0^c a(H+a)^{-(n+1)} da} = \frac{(n+1)\int_0^c a(H+a)^{-(n+2)} da}{\int_0^c a(H+a)^{-(n+1)} da}. \tag{53}$$

And

$$E_{f_3}(\theta|\underline{x}) = \frac{\int_0^\infty \theta\left[\int_0^c a^2\theta^n e^{-\theta(H+a)} da\right] d\theta}{\Gamma(n+1)\int_0^c a^2(H+a)^{-(n+1)} da} = \frac{(n+1)\int_0^c a^2(H+a)^{-(n+2)} da}{\int_0^c a^2(H+a)^{-(n+1)} da}. \tag{54}$$

Therefore, the hierarchical Bayesian estimates $\hat{\theta}_{HBB1}$, $\hat{\theta}_{HBB2}$ and $\hat{\theta}_{HBB3}$ can be obtained by using Eqs. (27), (52), (53) and (54) in Eq. (5) to be

$$\hat{\theta}_{HBB1} = \frac{wn}{H} + \frac{(1-w)(n+1)\int_0^c a(c-a)(H+a)^{-(n+2)} da}{\int_0^c a(c-a)(H+a)^{-(n+1)} da},$$

$$\hat{\theta}_{HBB2} = \frac{wn}{H} + \frac{(1-w)(n+1)\int_0^c a(H+a)^{-(n+2)} da}{\int_0^c a(H+a)^{-(n+1)} da}.$$

And

$$\hat{\theta}_{HBB3} = \frac{wn}{H} + \frac{(1-w)(n+1)\int_0^c a^2(H+a)^{-(n+2)} da}{\int_0^c a^2(H+a)^{-(n+1)} da}.$$

4.2 The hierarchical Bayesian estimation under PLF

Theorem 6. Assuming PLF in Eq. (6), the hierarchical posterior distributions in Eqs. (46), (47) and (48), then the hierarchical Bayesian estimates $\hat{\theta}_{HBP1}$, $\hat{\theta}_{HBP2}$ and $\hat{\theta}_{HBP3}$ of θ are the following:

$$\hat{\theta}_{HBP1} = \sqrt{\frac{(n+1)(n+2)\int_0^c a(c-a)(H+a)^{-(n+3)} da}{\int_0^c a(c-a)(H+a)^{-(n+1)} da}}, \tag{55}$$

$$\hat{\theta}_{HBP2} = \sqrt{\frac{(n+1)(n+2)\int_0^c a(H+a)^{-(n+3)}da}{\int_0^c a(H+a)^{-(n+1)}da}}\ . \tag{56}$$

And

$$\hat{\theta}_{HBP3} = \sqrt{\frac{(n+1)(n+2)\int_0^c a^2(H+a)^{-(n+3)}da}{\int_0^c a^2(H+a)^{-(n+1)}da}}\ . \tag{57}$$

Proof. We can derive $E_{f_i}(\theta^2 \,|\underline{x}), (i = 1,2,3)$ by using Eqs. (46), (47) and (48) to be

$$E_{f_1}(\theta^2 \,|\underline{x}) = \frac{\int_0^\infty \theta^2 \left[\int_0^c a(c-a)\theta^n e^{-\theta(H+a)}da\right]d\theta}{\Gamma(n+1)\int_0^c a(c-a)(H+a)^{-(n+1)}da} = \frac{(n+1)(n+2)\int_0^c a(c-a)(H+a)^{-(n+3)}da}{\int_0^c a(c-a)(H+a)^{-(n+1)}da}, \tag{58}$$

$$E_{f_2}(\theta^2 \,|\underline{x}) = \frac{\int_0^\infty \theta^2 \left[\int_0^c a\theta^n e^{-\theta(H+a)}da\right]d\theta}{\Gamma(n+1)\int_0^c a(H+a)^{-(n+1)}da} = \frac{(n+1)(n+20\int_0^c a(H+a)^{-(n+3)}da}{\int_0^c a(H+a)^{-(n+1)}da}\ . \tag{59}$$

And

$$E_{f_3}(\theta^2 \,|\underline{x}) = \frac{\int_0^\infty \theta^2 \left[\int_0^c a^2\theta^n e^{-\theta(H+a)}da\right]d\theta}{\Gamma(n+1)\int_0^c a^2(H+a)^{-(n+1)}da} = \frac{(n+1)(n+2)\int_0^c a^2(H+a)^{-(n+3)}da}{\int_0^c a^2(H+a)^{-(n+1)}da} \tag{60}$$

Therefore, the hierarchical Bayesian estimates $\hat{\theta}_{HBP1}$, $\hat{\theta}_{HBP2}$ and $\hat{\theta}_{HBP3}$ can be obtained by using Eqs. (58), (59) and (60) in Eq. (7) to be

$$\hat{\theta}_{HBP1} = \sqrt{\frac{(n+1)(n+2)\int_0^c a(c-a)(H+a)^{-(n+3)}da}{\int_0^c a(c-a)(H+a)^{-(n+1)}da}},$$

$$\hat{\theta}_{HBP2} = \sqrt{\frac{(n+1)(n+2)\int_0^c a(H+a)^{-(n+3)}da}{\int_0^c a(H+a)^{-(n+1)}da}}\ .$$

And

$$\hat{\theta}_{HBP3} = \sqrt{\frac{(n+1)(n+2)\int_0^c a^2(H+a)^{-(n+3)}da}{\int_0^c a^2(H+a)^{-(n+1)}da}}\ .$$

4.3 The hierarchical Bayesian estimation under ELF

Theorem 7. Assuming ELF in Eq. (8), the hierarchical posterior distributions in Eqs. (46), (47) and (48), then the hierarchical Bayesian estimates $\hat{\theta}_{HBE1}$, $\hat{\theta}_{HBE2}$ and $\hat{\theta}_{HBE3}$ of θ are the following:

$$\hat{\theta}_{HBE1} = \frac{n\int_0^c a(c-a)(H+a)^{-(n+1)}da}{\int_0^c a(c-a)(H+a)^{-n}\,da}, \tag{61}$$

$$\hat{\theta}_{HBE2} = \frac{n\int_0^c a(H+a)^{-(n+1)}da}{\int_0^c a(H+a)^{-n}\,da}. \tag{62}$$

And

$$\hat{\theta}_{HBE3} = \frac{n\int_0^c a^2(H+a)^{-(n+1)}da}{\int_0^c a^2(H+a)^{-n}\,da}. \tag{63}$$

Proof. We can derive $E_{f_i}(\theta^{-1}|\underline{x}),(i=1,2,3)$ by using (46), (47) and (48) to be

$$E_{f_1}(\theta^{-1}|\underline{x}) = \frac{\int_0^\infty \theta^{-1}\left[\int_0^c a(c-a)\theta^n e^{-\theta(H+a)}da\right]d\theta}{\Gamma(n+1)\int_0^c a(c-a)(H+a)^{-(n+1)}da} = \frac{\int_0^c a(c-a)(H+a)^{-n}\,da}{n\int_0^c a(c-a)(H+a)^{-(n+1)}da}, \tag{64}$$

$$E_{f_2}(\theta^{-1}|\underline{x}) = \frac{\int_0^\infty \theta^{-1}\left[\int_0^c a\theta^n e^{-\theta(H+a)}da\right]d\theta}{\Gamma(n+1)\int_0^c a(H+a)^{-(n+1)}da} = \frac{\int_0^c a(H+a)^{-n}\,da}{n\int_0^c a(H+a)^{-(n+1)}da}. \tag{65}$$

And

$$E_{f_3}(\theta^{-1}|\underline{x}) = \frac{\int_0^\infty \theta^{-1}\left[\int_0^c a^2\theta^n e^{-\theta(H+a)}da\right]d\theta}{\Gamma(n+1)\int_0^c a^2(H+a)^{-(n+1)}da} = \frac{\int_0^c a^2(H+a)^{-n}\,da}{n\int_0^c a^2(H+a)^{-(n+1)}da}. \tag{66}$$

Consequentely, the hierarchical Bayesian estimates $\hat{\theta}_{HBE1}$, $\hat{\theta}_{HBE2}$ and $\hat{\theta}_{HBE3}$ can be obtained by using Eqs. (64), (65) and (66) inEq. (9) to be

$$\hat{\theta}_{HBE1} = \frac{n\int_0^c a(c-a)(H+a)^{-(n+1)}da}{\int_0^c a(c-a)(H+a)^{-n}\,da},$$

$$\hat{\theta}_{HBE2} = \frac{n\int_0^c a(H+a)^{-(n+1)}da}{\int_0^c a(H+a)^{-n}\,da}.$$

And

$$\hat{\theta}_{HBE3} = \frac{n\int_0^c a^2 (H+a)^{-(n+1)} da}{\int_0^c a^2 (H+a)^{-n} da}.$$

4.4 The hierarchical Bayesian estimation under DLF

Theorem 8. Assuming DLF in Eq. (10), the hierarchical posterior distributions in Eqs. (46), (47) and (48), then the hierarchical Bayesian estimates $\hat{\theta}_{HBD1}$, $\hat{\theta}_{HBD2}$ and $\hat{\theta}_{HBD3}$ of θ are the following:

$$\hat{\theta}_{HBD1} = \frac{(n+2)\int_0^c a(c-a)(H+a)^{-(n+3)} da}{\int_0^c a(c-a)(H+a)^{-(n+2)} da}, \tag{67}$$

$$\hat{\theta}_{HBD2} = \frac{(n+2)\int_0^c a(H+a)^{-(n+3)} da}{\int_0^c a(H+a)^{-(n+2)} da}. \tag{68}$$

And

$$\hat{\theta}_{HBD3} = \frac{(n+2)\int_0^c a^2(H+a)^{-(n+3)} da}{\int_0^c a^2(H+a)^{-(n+2)} da}. \tag{69}$$

Proof. The hierarchical Bayesian estimates $\hat{\theta}_{HBD1}$ can be obtained by using Eqs. (52) and (58) in Eq. (11) to be

$$\hat{\theta}_{HBD1} = \frac{(n+1)(n+2)\int_0^c a(c-a)(H+a)^{-(n+3)} da \Big/ \int_0^c a(c-a)(H+a)^{-(n+1)} da}{(n+1)\int_0^c a(c-a)(H+a)^{-(n+2)} da \Big/ \int_0^c a(c-a)(H+a)^{-(n+1)} da}$$
$$= \frac{(n+2)\int_0^c a(c-a)(H+a)^{-(n+3)} da}{\int_0^c a(c-a)(H+a)^{-(n+2)} da}.$$

Similarly, the hierarchical Bayesian estimates $\hat{\theta}_{HBD2}$ and $\hat{\theta}_{HBD3}$ can be obtained by using Eqs. (53), (59) and (54), (60) in Eq. (11) to be

$$\hat{\theta}_{HBD2} = \frac{(n+1)(n+2)\int_0^c a(H+a)^{-(n+3)} da \Big/ \int_0^c a(H+a)^{-(n+1)} da}{(n+1)\int_0^c a(H+a)^{-(n+2)} da \Big/ \int_0^c a(H+a)^{-(n+1)} da} = \frac{(n+2)\int_0^c a(H+a)^{-(n+3)} da}{\int_0^c a(H+a)^{-(n+2)} da}.$$

And

$$\hat{\theta}_{HBD3} = \frac{(n+1)(n+2)\int_0^c a^2(H+a)^{-(n+3)} da \Big/ \int_0^c a^2(H+a)^{-(n+1)} da}{(n+1)\int_0^c a^2(H+a)^{-(n+2)} da \Big/ \int_0^c a^2(H+a)^{-(n+1)} da} = \frac{(n+2)\int_0^c a^2(H+a)^{-(n+3)} da}{\int_0^c a^2(H+a)^{-(n+2)} da}.$$

5 Properties of the E-Bayesian and Hierarchical Bayesian Estimates

In this section, we shall discussed the properties of E-Bayesian estimates and the relations among the E-Bayesian and hierarchical Bayesian estimates.

5.1 The relations between the E-Bayesian estimates

In this subsection, we will construct the relations between the E-Bayesian and the hierarchical Bayesian estimates.

5.1.1 Relations among $\hat{\theta}_{EBBi}\,(i=1,2,3)$

Lemma 1. It follows from Eqs. (21), (22) and (23) that

(i) $\hat{\theta}_{EBB3} < \hat{\theta}_{EBB2} < \hat{\theta}_{EBB1}$.

(ii) $\lim_{H\to\infty}\hat{\theta}_{EBB1} = \lim_{H\to\infty}\hat{\theta}_{EBB2} = \lim_{H\to\infty}\hat{\theta}_{EBB3}$.

Proof. See Appendix (1).

5.1.2 Relations among $\hat{\theta}_{EBPi}\,(i=1,2,3)$

Lemma 2. It follows from Eqs. (30), (31) and (32) that

(i) $\hat{\theta}_{EBP3} < \hat{\theta}_{EBP2} < \hat{\theta}_{EBP1}$.

(ii) $\lim_{H\to\infty}\hat{\theta}_{EBP1} = \lim_{H\to\infty}\hat{\theta}_{EBP2} = \lim_{H\to\infty}\hat{\theta}_{EBP3}$.

Proof. See Appendix (1).

5.1.3 Relations among $\hat{\theta}_{EBEi}\,(i=1,2,3)$

Lemma 3. It follows from Eqs. (35), (36) and (37) that

(i) $\hat{\theta}_{EBE3} < \hat{\theta}_{EBE2} < \hat{\theta}_{EBE1}$.

(ii) $\lim_{H\to\infty}\hat{\theta}_{EBE1} = \lim_{H\to\infty}\hat{\theta}_{EBE2} = \lim_{H\to\infty}\hat{\theta}_{EBE3}$.

Proof. See Appendix (1).

5.1.4 Relations among $\hat{\theta}_{EBDi}\,(i=1,2,3)$

Lemma 4. It follows from Eqs. (40), (41) and (42) that

(i) $\hat{\theta}_{EBD3} < \hat{\theta}_{EBD2} < \hat{\theta}_{EBD1}$.

(ii) $\lim_{H\to\infty}\hat{\theta}_{EBD1} = \lim_{H\to\infty}\hat{\theta}_{EBD2} = \lim_{H\to\infty}\hat{\theta}_{EBD3}$.

Proof. See Appendix (1).

5.2 The relations between theE-Bayesian and hierarchical Bayesian estimates

In this subsection, we will construct the relations between the E-Bayesian and the hierarchical Bayesian estimates .

5.2.1 Relations among $\hat{\theta}_{EBBi}, \hat{\theta}_{HBBi}\ (i=1,2,3)$

Lemma 5. It follows from Eqs. (21), (22), (23), (49), (50) and (51) that

$$\lim_{H\to\infty}\hat{\theta}_{EBBi}=\lim_{H\to\infty}\hat{\theta}_{HBBi}\ (i=1,2,3)\ .$$

Proof. See Appendix (2).

5.2.2 Relations among $\hat{\theta}_{EBPi}, \hat{\theta}_{HBPi}\ (i=1,2,3)$

Lemma 6. It follows from Eqs. (30), (31), (32), (55), (56) and (57) that

$$\lim_{H\to\infty}\hat{\theta}_{EBPi}=\lim_{H\to\infty}\hat{\theta}_{HBPi}\ (i=1,2,3)\ .$$

Proof. See Appendix (2).

5.2.3 Relations among $\hat{\theta}_{EBEi}, \hat{\theta}_{HBEi}\ (i=1,2,3)$

Lemma 7. It follows from Eqs. (35), (36), (37), (61), (62) and (63) that

$$\lim_{H\to\infty}\hat{\theta}_{EBEi}=\lim_{H\to\infty}\hat{\theta}_{HBEi}\ (i=1,2,3)\ .$$

Proof. See Appendix (2).

5.2.4 Relations among $\hat{\theta}_{EBDi}, \hat{\theta}_{HBDi}\ (i=1,2,3)$

Lemma 8. It follows from Eqs. (40), (41), (42), (67), (67) and (69) that

$$\lim_{H\to\infty}\hat{\theta}_{EBDi}=\lim_{H\to\infty}\hat{\theta}_{HBDi}\ (i=1,2,3)\ .$$

Proof. See Appendix (2).

6 Numerical Computations

The lower record values can be derived from the **DGOS** as a special case by taking $m=-1$ and $k=1$. Consequentely, the resulting estimates obtained in the ealiar sections can be specialized to lower records. The E-Bayesian and hierarical Bayesian estimates of θ are computed and compared based on a Monte Carlo simulation study described in the following steps:

Step (1): Set the default values (true values) of λ, c and w which are 3, 6 and 0.4 respectively. We used different sample sizes to investigate their effects on the resulting estimates.

Step (2): Based on these cases, we generate a from the uniform hyperprior distributions $(0,c)$ given in Eqs. (18), (19) and (20). For given values of a , we generate θ from the expoential distribution given in Eq. (15).

Step (3): Based on known values of λ samples are generated from the IWD distribution given in Eqs. (1) and (2).

Step (4): Computing the E-Bayesian and hierarchical Bayesian estimates of θ associated to the IWD according to formulas that have been obtained.

Step (5): We repeated this process 10000 times and compute the absolute bias (ABias) and mean square error (MSE) for the estimates for different sample sizes and given values of λ, c and w .

Where

$$ABias(\hat{\theta}) = \left|\hat{\theta} - \theta\right| \qquad MSE(\hat{\theta}) = \frac{1}{10000}\sum(\hat{\theta} - \theta)^2$$

and $\hat{\theta}$ stands for an estimator of θ . The simulation results are viewed in Tables 1-4.

Table 1. Averaged values of ABias and MSEs for estimates of the parameter θ based on BSELF

n	$\hat{\theta}_{EBB}$		$\hat{\theta}_{HBB}$	
	ABias	**MSE**	**ABias**	**MSE**
15	1.0070028	0.4796970	1.0751591	0.5502426
	0.9732256	0.4470480	1.0748000	0.5498963
	0.9394484	0.4155747	1.0679751	0.5429536
25	0.7437928	0.2640674	0.7956031	0.3044828
	0.7168066	0.2445895	0.7950753	0.3041068
	0.6898204	0.2258747	0.7877187	0.2985288
35	0.5613121	0.1523152	0.6013551	0.1765609
	0.5391035	0.1400339	0.6005888	0.1761497
	0.5168950	0.1282810	0.5926188	0.1715355
50	0.4177900	0.0856861	0.4479716	0.0997884
	0.3994477	0.0780095	0.4468238	0.0993307
	0.3811053	0.0707016	0.4380879	0.0955187
70	0.3175326	0.0506737	0.3405479	0.0592816
	0.3018943	0.0455763	0.3388726	0.0587725
	0.2862559	0.0407550	0.3293457	0.0555444
100	0.2100934	0.0234358	0.2248963	0.0275933
	0.3073146	0.0205514	0.2221054	0.0270156
	0.1847028	0.0178597	0.2116522	0.0245279

Table 2. Averaged values of ABias and MSEs for estimates of the parameter θ based on PLF

n	$\hat{\theta}_{EBP}$		$\hat{\theta}_{HBP}$	
	ABias	**MSE**	**ABias**	**MSE**
15	0.9622578	0.4360601	1.0765185	0.5516179
	0.9056845	0.3847179	1.0759232	0.5510431
	0.8491112	0.3366737	1.0645482	0.5394842
25	0.7100372	0.2392957	0.7971523	0.3056475
	0.6647446	0.2087013	0.7962794	0.3050247
	0.6194520	0.1802565	0.7840181	0.2957424
35	0.5355267	0.1376350	0.6031474	0.1775850
	0.4981513	0.1183598	0.6018844	0.1769053
	0.4607759	0.1005811	0.5886000	0.1692320
50	0.3987801	0.0773118	0.4501188	0.1007058
	0.3677878	0.0652576	0.4482361	0.0999520
	0.3367955	0.0542561	0.4336701	0.0936183
70	0.3035900	0.0457069	0.3431674	0.0601369
	0.2770295	0.0376892	0.3404335	0.0593016
	0.2504691	0.0304683	0.3245247	0.0539407
100	0.2023822	0.0212342	0.2286103	0.0284034
	0.1805722	0.0166678	0.2240774	0.0274569
	0.1587622	0.0126705	0.2064656	0.0233119

Table 3. Averaged values of ABias and MSEs for estimates of the parameter θ based on ELF

n	$\hat{\theta}_{EBE}$		$\hat{\theta}_{HBE}$	
	ABias	**MSE**	**ABias**	**MSE**
15	0.9472240	0.4225737	1.0594803	0.5343403
	0.8914860	0.3727779	1.0588753	0.5337653
	0.8357481	0.3261836	1.0475005	0.5223903
25	0.6939879	0.2286402	0.7788035	0.2917947
	0.6496444	0.1993613	0.7779096	0.2911717
	0.6053009	0.1721429	0.7656486	0.2821046
35	0.5183290	0.1289875	0.5832928	0.1661569
	0.4820405	0.1108701	0.5819857	0.1654769
	0.4457520	0.0941635	0.5687033	0.1580592
50	0.3801373	0.0703154	0.4283531	0.0912912
	0.3504159	0.0592907	0.4263744	0.0905373
	0.3206946	0.0492343	0.4118257	0.0845190
70	0.2833044	0.0398852	0.3192074	0.0521472
	0.2582429	0.0328181	0.3162854	0.0513155
	0.2331814	0.0264604	0.3004762	0.0463580
100	0.1790140	0.0167392	0.2005630	0.0220308
	0.1591776	0.4225737	1.0594803	0.5343403
	0.1393412	0.3727779	1.0588753	0.5337653

Table 4. Averaged values of ABias and MSEs for estimates of the parameter θ based on DLF

n	$\hat{\theta}_{EBD}$		$\hat{\theta}_{HBD}$	
	ABias	**MSE**	**ABias**	**MSE**
15	0.9672856	0.4406176	1.0822170	0.5574578
	0.9104329	0.3887529	1.0816249	0.5568831
	0.8535802	0.3402190	1.0702501	0.5452629
25	0.7154120	0.2429184	0.8032981	0.3103592
	0.6698015	0.2118770	0.8024321	0.3097365
	0.6241911	0.1830155	0.7901710	0.3003825
35	0.5412964	0.1405991	0.6098057	0.1815054
	0.5035564	0.1209274	0.6085620	0.1808260
	0.4658163	0.1027816	0.5952779	0.1730674
50	0.4050512	0.0797400	0.4574452	0.1039787
	0.3736314	0.0673291	0.4555941	0.1032255
	0.3422116	0.0560003	0.4410254	0.0967866
70	0.3104372	0.0477616	0.3512663	0.0629663
	0.2833708	0.0394097	0.3485943	0.0621311
	0.2563044	0.0318851	0.3326605	0.0566357
100	0.2103298	0.0228854	0.2381894	0.0307656
	0.9672856	0.0179989	0.2337885	0.0298118
	0.9104329	0.0137170	0.2159957	0.0254559

7 Conclusion Remarks

The E-Bayesian and hierarchical Bayesian estimates of the scale parameter of IWD are computed based on **DGOS**. The results are specialized to the lower record values. It has been noticed, from Tables 1-4, that the ABias and MSE of the resulting estimates decreses as the sample size increases. Numerical computations showed that the E-Bayesian estimates have smaller ABias and MSE than the hierarchical Bayesian estimates based on various loss functions and different sample sizes. Furthermore, in comparing the E-Bayesian estimates under different loss functions, we can deduct that the E-Bayesian estimates based on ELF are the most efficient whearse the E-Bayesian estimates based on BSELF are the least efficient in all cases. Finally, this work is showed that E-Bayesian criteria can provide more efficient estimates than the hierarchical Bayesian approach under **DGOS**.

Competing Interests

Authors have declared that no competing interests exist.

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Appendix 1

Proof of Lemma 1.

(i) From Eqs. (22), (23) and (24), we get

$$\hat{\theta}_{EBB2} - \hat{\theta}_{EBB3} = \hat{\theta}_{EBB1} - \hat{\theta}_{EBB2} = \left(\frac{(1-w)(n+1)}{c}\right)\left[\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)-2\right]. \quad \text{(A.1)}$$

For $-1 < x < 1$, we have: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$.

Assuming $x = \frac{c}{H}$ when $0 < c < H$, $0 < \frac{c}{H} < 1$, we get

$$\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)-2 = \left(1+\frac{2H}{c}\right)\left[\frac{c}{H} - \frac{1}{2}\left(\frac{c^2}{H^2}\right) + \frac{1}{3}\left(\frac{c^3}{H^3}\right) - \frac{1}{4}\left(\frac{c^4}{H^4}\right) + \frac{1}{5}\left(\frac{c^5}{H^5}\right) - \frac{1}{6}\left(\frac{c^6}{H^6}\right) + ...\right] - 2$$

$$= \left(\frac{2}{3}-\frac{1}{2}\right)\left(\frac{c^2}{H^2}\right) + \left(\frac{1}{3}-\frac{2}{4}\right)\left(\frac{c^3}{H^3}\right) + \left(\frac{2}{5}-\frac{1}{4}\right)\left(\frac{c^4}{H^4}\right) + \left(\frac{1}{5}-\frac{2}{6}\right)\left(\frac{c^5}{H^5}\right) + ...$$

$$= \left[\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right) + \frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right) + ...\right] > 0 . \quad \text{(A.2)}$$

According to Eqs.(A.1) and (A.2), we have

$$\hat{\theta}_{EBB2} - \hat{\theta}_{EBB3} = \hat{\theta}_{EBB1} - \hat{\theta}_{EBB2} > 0$$

That is $\hat{\theta}_{EBB3} < \hat{\theta}_{EBB2} < \hat{\theta}_{EBB1}$.

(ii) From Eqs. (A.1) and (A.2), we get

$$\lim_{H\to\infty}\left(\hat{\theta}_{EBB2} - \hat{\theta}_{EBB3}\right) = \lim_{H\to\infty}\left(\hat{\theta}_{EBB1} - \hat{\theta}_{EBB2}\right)$$
$$= \left(\frac{(1-w)(n+1)}{c}\right)\lim_{H\to\infty}\left[\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right) + \frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right) + ...\right] = 0 . \quad \text{(A.3)}$$

That is $\lim_{H\to\infty}\hat{\theta}_{EBB1} = \lim_{H\to\infty}\hat{\theta}_{EBB2} = \lim_{H\to\infty}\hat{\theta}_{EBB3}$.

Proof of Lemma 2.

(i) From Eqs. (30), (31) and (32), we get

$$\hat{\theta}_{EBP2} - \hat{\theta}_{EBP3} = \hat{\theta}_{EBP1} - \hat{\theta}_{EBP2} = \left(\frac{\sqrt{(n+1)(n+2)}}{c}\right)\left[\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)-2\right]. \quad \text{(A.4)}$$

According to Eqs.(A.2) and (A.4), we have

$$\hat{\theta}_{EBP2} - \hat{\theta}_{EBP3} = \hat{\theta}_{EBP1} - \hat{\theta}_{EBP2} > 0 .$$

That is $\hat{\theta}_{EBP3} < \hat{\theta}_{EBP2} < \hat{\theta}_{EBP1}$.

(ii) From Eqs. (A.3) and (A.4), we get

$$\lim_{H\to\infty}\left(\hat{\theta}_{EBP2} - \hat{\theta}_{EBP3}\right) = \lim_{H\to\infty}\left(\hat{\theta}_{EBP1} - \hat{\theta}_{EBP2}\right)$$

$$= \left(\frac{\sqrt{(n+1)(n+2)}}{c}\right)\lim_{H\to\infty}\left[\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)+\frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right)+...\right]=0 \qquad \text{(A.5)}$$

That is $\lim_{H\to\infty}\hat{\theta}_{EBP1} = \lim_{H\to\infty}\hat{\theta}_{EBP2} = \lim_{H\to\infty}\hat{\theta}_{EBP3}$.

Proof of Lemma 3.

(i) From Eqs. (35), (36) and (37), we obtain

$$\hat{\theta}_{EBE2} - \hat{\theta}_{EBE3} = \hat{\theta}_{EBE1} - \hat{\theta}_{EBE2} = \left(\frac{n}{c}\right)\left[\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)-2\right]. \qquad \text{(A.6)}$$

According to Eqs.(A.2) and (A.6), we have

$$\hat{\theta}_{EBE2} - \hat{\theta}_{EBE3} = \hat{\theta}_{EBE1} - \hat{\theta}_{EBE2} > 0 .$$

That is $\hat{\theta}_{EBE3} < \hat{\theta}_{EBE2} < \hat{\theta}_{EBE1}$.

(ii) From Eqs. (A.3) and (A.6), we get

$$\lim_{H\to\infty}\left(\hat{\theta}_{EBE2} - \hat{\theta}_{EBE3}\right) = \lim_{H\to\infty}\left(\hat{\theta}_{EBE1} - \hat{\theta}_{EBE2}\right) = \left(\frac{n}{c}\right)\lim_{H\to\infty}\left[\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)+\frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right)+...\right]=0 \qquad \text{(A.7)}$$

That is $\lim_{H\to\infty}\hat{\theta}_{EBE1} = \lim_{H\to\infty}\hat{\theta}_{EBE2} = \lim_{H\to\infty}\hat{\theta}_{EBE3}$.

Proof of Lemma 4.

(i) FromEqs. (40), (41) and (42) that

$$\hat{\theta}_{EBD2} - \hat{\theta}_{EBD3} = \hat{\theta}_{EBD1} - \hat{\theta}_{EBD2} = \left(\frac{n+2}{c}\right)\left[\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)-2\right]. \qquad \text{(A.8)}$$

According to Eqs.(A.2) and (A.8), we have

$$\hat{\theta}_{EBD2} - \hat{\theta}_{EBD3} = \hat{\theta}_{EBD1} - \hat{\theta}_{EBD2} > 0 .$$

That is $\hat{\theta}_{QEBD3} < \hat{\theta}_{QEBD2} < \hat{\theta}_{QEBD1}$.

(ii) From Eqs. (A.3) and (A.8), we get

$$\lim_{H\to\infty}\left(\hat{\theta}_{EBD2}-\hat{\theta}_{EBD3}\right)=\lim_{H\to\infty}\left(\hat{\theta}_{EBD1}-\hat{\theta}_{EBD2}\right)$$

$$=\left(\frac{n+2}{c}\right)\lim_{H\to\infty}\left[\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)+\frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right)+\ldots\right]=0\,. \quad (A.9)$$

That is $\lim_{H\to\infty}\hat{\theta}_{EBD1}=\lim_{H\to\infty}\hat{\theta}_{EBD2}=\lim_{H\to\infty}\hat{\theta}_{EBD3}$.

Appendix 2

Proof of Lemma 5.

Since, $a(c-a)$, $\frac{1}{(a+H)^{n+2}}$ are continuous on $(0,c)$, based on the extended case of mean value theorem for definite integrals [when $0<a<c, a(c-a)>0$], there is as least one number $a_1 \in (0,c)$ such that

$$\int_0^c \frac{a(c-a)da}{(a+H)^{n+2}} = \frac{1}{(a_1+H)^{n+2}} \int_0^c a(c-a)da = \frac{c^3}{6(a_1+H)^{n+2}} . \tag{B.1}$$

Similarly, there is as least one number $a_2 \in (0,c)$ such that

$$\int_0^c \frac{a(c-a)da}{(a+H)^{n+1}} = \frac{1}{(a_2+H)^{n+1}} \int_0^c a(c-a)da = \frac{c^3}{6(a_2+H)^{n+1}} . \tag{B.2}$$

By substitution from Eqs.(B.1) and (B.2) in Eq. (49), we get

$$\hat{\theta}_{HBB1} = \frac{wn}{H} + \left(\frac{(1-w)(n+1)}{a_1+H}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n+1} . \tag{B.3}$$

By taking the limit as H tends to ∞ for both sides of Eq. (B.3), we obtain

$$\lim_{H\to\infty} \hat{\theta}_{HBB1} = \lim_{H\to\infty} \left\{\frac{wn}{H} + \left(\frac{(1-w)(n+1)}{a_1+H}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n+1}\right\} = 0 \quad . \tag{B.4}$$

According to Eqs.(A.3) and (B.4), we can deduct that

$$\lim_{H\to\infty} \hat{\theta}_{EBB1} = \lim_{H\to\infty} \hat{\theta}_{HBB1} .$$

Also a, $\frac{1}{(a+H)^{n+2}}$ are continuous on $(0,c)$, according to the extended case of mean value theorem for definite integrals (when $0<a<c$), there is as least one number $a_1 \in (0,c)$ such that

$$\int_0^c \frac{a\,da}{(a+H)^{n+2}} = \frac{1}{(a_1+H)^{n+2}} \int_0^c a da = \frac{c^2}{2(a_1+H)^{n+2}} . \tag{B.5}$$

Similarly, there is as least one number $a_2 \in (0,c)$ such that

$$\int_0^c \frac{a\,da}{(a+H)^{n+1}} = \frac{1}{(a_2+H)^{n+1}} \int_0^c a da = \frac{c^2}{2(a_2+H)^{n+1}} . \tag{B.6}$$

By substitution from Eqs.(B.5) and (B.6) in Eq. (50), we get

$$\hat{\theta}_{HBB2} = \frac{wn}{H} + \left(\frac{(1-w)(n+1)}{a_1+H}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n+1} . \tag{B.7}$$

By taking the limit as H tends to ∞ for both sides of Eq. (B.7), we obtain

$$\lim_{H\to\infty}\hat{\theta}_{HBB\,2} = \lim_{H\to\infty}\left\{\frac{wn}{H}+\left(\frac{(1-w)(n+1)}{a_1+H}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n+1}\right\}=0 \quad . \tag{B.8}$$

According to Eqs.(A.3) and (B.8), we can deduct that

$$\lim_{H\to\infty}\hat{\theta}_{EBB\,2} = \lim_{H\to\infty}\hat{\theta}_{HBB\,2}\,.$$

Furthermore a^2, $\frac{1}{(a+H)^{n+2}}$ are continuous on $(0,c)$, based on the extended case of mean value theorem for definite integrals (when $0<a<c$), there is as least one number $a_1\in(0,c)$ such that

$$\int_0^c \frac{a^2\,da}{(a+H)^{n+2}} = \frac{1}{(a_1+H)^{n+2}}\int_0^c a^2 da = \frac{3c^2}{3(a_1+H)^{n+2}}\,. \tag{B.9}$$

Similarly, there is as least one number $a_2\in(0,c)$ such that

$$\int_0^c \frac{a^2\,da}{(a+H)^{n+1}} = \frac{1}{(a_2+H)^{n+1}}\int_0^c a^2 da = \frac{c^2}{3(a_2+H)^{n+1}}\,. \tag{B.10}$$

By substitution from Eqs.(B.9) and (B.10) in Eq. (51), we get

$$\hat{\theta}_{HBB\,3} = \frac{wn}{H}+\left(\frac{(1-w)(n+1)}{a_1+H}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n+1}. \tag{B.11}$$

By taking the limit as H tends to ∞ for both sides of Eq. (B.11), we obtain

$$\lim_{H\to\infty}\hat{\theta}_{HBB\,3} = \lim_{H\to\infty}\left\{\frac{wn}{H}+\left(\frac{(1-w)(n+1)}{a_1+H}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n+1}\right\}=0 \quad . \tag{B.12}$$

According to Eqs. (A.3) and (B.12), we can deduct that

$$\lim_{H\to\infty}\hat{\theta}_{EBB\,3} = \lim_{H\to\infty}\hat{\theta}_{HBB\,3}\,.$$

Proof of Lemma 6.

By using similar steps in lemma (5), we can get

$$\int_0^c \frac{a(c-a)da}{(a+H)^{n+3}} = \frac{1}{(a_1+H)^{n+3}}\int_0^c a(c-a)da = \frac{c^3}{6(a_1+H)^{n+3}}\,. \tag{B.13}$$

By substitution from Eqs.(B.2) and (B.12) in Eq. (55), we get

$$\hat{\theta}_{HBP1} = \sqrt{\left(\frac{(n+1)(n+2)}{(a_1+H)^2}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n+1}} \,. \tag{B.14}$$

By taking the limit as H tends to ∞ for both sides of Eq. (B.14), we obtain

$$\lim_{H\to\infty}\hat{\theta}_{HBP1} = \lim_{H\to\infty}\sqrt{\left(\frac{(n+1)(n+2)}{(a_1+H)^2}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n+1}} = 0\,. \tag{B.15}$$

According to Eqs.(A.5) and (B.15), we can deduct that

$$\hat{\theta}_{EBP1} = \hat{\theta}_{HBP1}\,.$$

Similarly, we can obtain

$$\lim_{H\to\infty}\hat{\theta}_{EBPi} = \lim_{H\to\infty}\hat{\theta}_{HBPi}\,(i=2,3)\,.$$

Proof of Lemma 7.

By using similar steps in lemma (5), we can get

$$\int_0^c \frac{a(c-a)da}{(a+H)^{n+1}} = \frac{1}{(a_1+H)^{n+1}}\int_0^c a(c-a)da = \frac{c^3}{6(a_1+H)^{n+1}}, \tag{B.16}$$

$$\int_0^c \frac{a(c-a)da}{(a+H)^{n}} = \frac{1}{(a_1+H)^{n}}\int_0^c a(c-a)da = \frac{c^3}{6(a_1+H)^{n}}\,. \tag{B.17}$$

By substitution from Eqs.(B.16) and (B.17) in Eq. (61), we get

$$\hat{\theta}_{HBE1} = \left(\frac{n}{a_1+H}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n}. \tag{B.18}$$

By taking the limit as H tends to ∞ for both sides of Eq. (B.18), we obtain

$$\lim_{H\to\infty}\hat{\theta}_{HBE1} = \lim_{H\to\infty}\left\{\left(\frac{n}{a_1+H}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n}\right\} = 0\,. \tag{B.19}$$

According toEqs.(A.7) and (B.19), we can deduct that

$$\lim_{H\to\infty}\hat{\theta}_{EBE1} = \lim_{H\to\infty}\hat{\theta}_{HBE1}\,.$$

Similarly, we can obtain

$$\lim_{H\to\infty}\hat{\theta}_{EBEi} = \lim_{H\to\infty}\hat{\theta}_{HBEi}\,(i=2,3)\,.$$

Proof of Lemma 8.

Bysubsitituting from Eqs. (B.1) and (B.13) in Eq. (67), we get

$$\hat{\theta}_{HBD1} = \left(\frac{n+2}{a_1+H}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n+1}. \tag{B.20}$$

By taking the limit as H tends to ∞ for both sides of Eq. (B.20), we obtain

$$\lim_{H\to\infty}\hat{\theta}_{HBD1} = \lim_{H\to\infty}\left\{\left(\frac{n+2}{a_1+H}\right)\left[\frac{a_2+H}{a_1+H}\right]^{n+1}\right\} = 0. \tag{B.21}$$

According to Eqs.(A.9) and (B.21), we can deduct that

$$\lim_{H\to\infty}\hat{\theta}_{EBD1} = \lim_{H\to\infty}\hat{\theta}_{HBD1}.$$

Similarly, we can obtain

$$\lim_{H\to\infty}\hat{\theta}_{EBDi} = \lim_{H\to\infty}\hat{\theta}_{HBdi}\ (i = 2,3).$$

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