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On a Discrete Time Semi-Markov Risk Model with Dividends and Stochastic Premiums

Cui Wang[1*]

[1]*School of Mathematics, Liaoning Normal University, Dalian 116029, China.*

Author's contribution

Cui Wang designed research, performed research, analyzed data, and wrote the paper.

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Abstract

A discrete semi-Markov risk model with dividends and stochastic premiums is investigated. We derive recursive equations for the expected penalty function by using the technique of probability generating function. Finally, a numerical example is given to illustrate the applicability of the results obtained.

Keywords: Discrete time semi-Markov risk model; stochastic premiums; randomized dividends; expected penalty function.

1 Introduction

The classical discrete time semi-Markov risk model is first introduced by Reinhard and Snoussi [1, 2], and defined as follows: Assume that $(J_n, n \in \mathbb{N})$ is a homogeneous, irreducible and aperiodic Markov chain with finite state space $M = \{1, \ldots, m\}$ $(m \in \mathbb{N}^+)$, which influences the distribution of the claims in each period. The one-step transition probability matrix $\mathbf{P} = (p_{ij})_{i,j \in M}$ is defined

*Corresponding author: E-mail: 584511477@qq.com;

as follows:

$$p_{ij} = \mathbb{P}(J_n = j | J_{n-1} = i, J_k, k \leq n-1),$$

and the unique stationary distribution is $\pi = (\pi_1, \ldots, \pi_m)$. The surplus process Z_t is described as:

$$Z_t = Z_{t-1} + 1 - Y_t, \quad t \in \mathbb{N}^+, \tag{1.1}$$

where Y_t denotes the total amount of claims for the t-th period and the premium for each period is 1. Assume that $\{Y_n, n \in \mathbb{N}^+\}$ is a sequence of non-negative integer random variables, conditionally independent given the Markov chain $(J_n, n \in \mathbb{N})$. Suppose that (J_t, Y_t) depends on $\{J_k, X_k; k \leq t-1\}$ only through J_{t-1}. Let

$$g_{ij}(l) = \mathbb{P}(Y_t = l, J_t = j | J_{t-1} = i, J_k, Y_k, k \leq t-1), \quad l \in \mathbb{N}, \tag{1.2}$$

and

$$\mu_{ij} = \sum_{k=0}^{\infty} k g_{ij}(k) < \infty, \ \mu_i = \sum_{j=1}^{m} \mu_{ij}, \quad i \in M.$$

For the discrete time semi-Markov risk model (1.1), Reinhard and Snoussi [1][2] derived recursive formula for the distribution of the surplus just prior to ruin and that of the deficit at ruin with some restrictions imposed on the total claim size. Chen et al. [3][4] relaxed the restriction of Reinhard and Snoussi and derived recursive formula for computing the expected discounted dividends and survival probabilities. Recently, Yuen et al. [5] further incorporated randomized dividends into risk model (1.1) and investigated the discounted Gerber-Shiu penalty function. Dividend strategies for insurance risk models were first proposed by De Finetti [6] to reflect more realistically the surplus cash flows in an insurance portfolio. Barrier strategies for the classical compound Poisson risk model have been studied extensively in the literature, see lin et al. [7] and the references therein. Randomized dividend strategies were proposed by Tan and Yang [8] for a compound binomial model, in which the insurer will pay a dividend of 1 with a probability in each time period if the surplus is greater than or equal to a non-negative integer at the beginning of the period. See Landriault [9], He and Yang [10], and Yuen et al. [11] for some generalizations.

In the present paper we show that the technique used in Yuen et al. [5] can be further generalized to the case with stochastic premiums. The motivation for the stochastic premiums is that the insurance company may have lump sums of income. To the best of our knowledge, Boucherie et al. [12] first added a compound Poisson process with positive jumps to the classical Cramér-Lundberg risk model to describe the stochastic income. Similar topics were discussed by many authors, see [13][14][15][16][17] for more details. More precisely, we consider the following surplus process

$$U_t = u + \sum_{i=1}^{t} \zeta_i - \sum_{i=1}^{t} Y_i - \sum_{i=1}^{t} \gamma_i 1_{(u_{i-1} \geq x)}, \quad t \in \mathbb{N}, \tag{1.3}$$

where the threshold value $x \in \mathbb{N}$, claim sizes $\{Y_i, i \in \mathbb{N}^+\}$ depend on the Markov chain $(J_n, n \in \mathbb{N})$ through the relationship (1.2). $\{\zeta_i, i \in \mathbb{N}\}$ and $\{\gamma_i, i \in \mathbb{N}\}$ are two i.i.d. Bernoulli sequences with $P(\zeta_i = 1) = p = 1 - q > 0$ and $P(\gamma_i = 0) = \alpha > 0$ respectively.

The ruin time for risk model (1.3) is defined as $\tau = \inf\{t \in \mathbb{N} : U_t < 0\}$ with $\tau = \infty$ if ruin does not occur. Then the Gerber-Shiu discounted free penalty function can be defined as

$$m_i(u) = E(\omega(U_{\tau^-}, |U_\tau|) 1_{(\tau < \infty)} | U_0 = u, J_0 = i), \quad i \in M, u \in \mathbb{N}, \tag{1.4}$$

where $\omega(x, y)$ is a non-negative bounded function. Obviously, when $\omega(x, y) \equiv 1$, equation (1.4) reduces to be the ruin probability $\psi_i(u) = P(\tau < \infty | U_0 = u, J_0 = i)$, the corresponding survival probability is denoted by $\phi_i(u) = 1 - \psi_i(u)$. To ensure that ruin will not almost surely occur, we assume positive safety load condition $\sum_{i=1}^{m} \pi_i \mu_i < p + \alpha - 1 = \alpha - q$ always holds.

The rest of the paper is structured as follows. In section 2, we analyze the recursive equations satisfied by the expected discounted free penalty function. Finally, numerical illustrations are given in Section 3.

2 Recursive Equations for Expected Discounted Free Penalty Function

In this section, we derive recursive equations satisfied by $m_i(u)$ using the technique proposed in [5] with some modifications. We only consider the special case of $m = 2$. By considering the occurrence (or not) of the premium income and claims in the next period, for $0 \leq u < x$, the total probability formula implies that

$$m_i(u) = p[\sum_{j=1}^{2}\sum_{k=0}^{u+1} g_{ij}(k)m_j(u+1-k) + \xi_i(u)] + q[\sum_{j=1}^{2}\sum_{k=0}^{u} g_{ij}(k)m_j(u-k) + \eta_i(u)], \tag{2.1}$$

when $u \geq x$,

$$\begin{aligned} m_i(u) = {} & \alpha\{p[\sum_{j=1}^{2}\sum_{k=0}^{u+1} g_{ij}(k)m_j(u+1-k) + \xi_i(u)] + q[\sum_{j=1}^{2}\sum_{k=0}^{u} g_{ij}(k)m_j(u-k) + \eta_i(u)]\} \\ & + (1-\alpha)\{p[\sum_{j=1}^{2}\sum_{k=0}^{u} g_{ij}(k)m_j(u-k) + \eta_i(u)] + q[\sum_{j=1}^{2}\sum_{k=0}^{u-1} g_{ij}(k)m_j(u-1-k) + \zeta_i(u)]\}, \end{aligned} \tag{2.2}$$

where

$$g_i(k) = \sum_{j=1}^{2} g_{ij}(k), \qquad \xi_i(u) = \sum_{k=u+2}^{\infty} g_i(k)\omega(u+1, k-u-1),$$

$$\eta_i(u) = \sum_{k=u+1}^{\infty} g_i(k)\omega(u, k-u), \qquad \zeta_i(u) = \sum_{k=u}^{\infty} g_i(k)\omega(u-1, k-u+1), \quad i = 1, 2.$$

Let $\widetilde{m}_i(s)$ and $\widetilde{g}_{ij}(s)$ be the probability generating function of $m_i(u)$ and $g_{ij}(u)$. Multiplying both sides of the equation (2.1)(2.2) and summing over u from 0 to ∞, after some algebras we have

$$\begin{aligned} s\widetilde{m}_i(s) = {} & [\alpha + (1-\alpha)s][p+qs]\sum_{j=1}^{2}\widetilde{g}_{ij}(s)\widetilde{m}_j(s) + \alpha sp\widetilde{\xi}_i(s) + [\alpha sq + (1-\alpha)sp]\widetilde{\eta}_i(s) \\ & + (1-\alpha)sq\widetilde{\zeta}_i(s) - \alpha p\sum_{j=1}^{2} g_{ij}(0)m_j(0) + (1-\alpha)\sum_{u=0}^{x-1} N_i(u)s^{u+1}, \end{aligned}$$

where

$$\begin{aligned} N_i(u) = {} & p\sum_{j=1}^{2}\sum_{k=0}^{u+1} g_{ij}(k)m_j(m+1-k) + (q-p)\sum_{j=1}^{2}\sum_{k=0}^{u} g_{ij}(k)m_j(u-k) + p\xi_i(u) \\ & + (q-p)\eta_i(u) - q\zeta_i(u) - q\sum_{j=1}^{2}\sum_{k=0}^{u-1} g_{ij}(k)m_j(u-1-k), \; u = 0, 1, 2, \cdots, x-1. \end{aligned}$$

For $i = 1, 2$, let

$$A(\alpha, s) = \alpha + (1-\alpha)s, \quad e_i = \sum_{j=1}^{2} g_{ij}(0)m_j(0), \quad M_i(s) = \sum_{u=0}^{x-1} N_i(u)s^{u+1},$$

$$H_i(s) = \alpha pe_i - \alpha sp\widetilde{\xi}_i(s) - (1-\alpha)[sp\widetilde{\eta}_i(s) + M_i(s)] - \alpha sq\widetilde{\eta}_i(s) - (1-\alpha)sq\widetilde{\zeta}_i(s).$$

Then we have

$$\begin{cases} [(p+qs)A(\alpha,s)\widetilde{g}_{11}(s) - s]\widetilde{m}_1(s) + (p+qs)A(\alpha,s)\widetilde{g}_{12}(s)\widetilde{m}_2(s) = H_1(s), \\ (p+qs)A(\alpha,s)\widetilde{g}_{21}(s)\widetilde{m}_1(s) + [(p+qs)A(\alpha,s)\widetilde{g}_{22}(s) - s]\widetilde{m}_2(s) = H_2(s). \end{cases} \tag{2.3}$$

From (2.3) we get

$$\begin{aligned} &\{[(p+qs)A(\alpha,s)\widetilde{g}_{11}(s) - s][(p+qs)A(\alpha,s)\widetilde{g}_{22}(s) - s] - (p+qs)^2A^2(\alpha,s)\widetilde{g}_{12}(s)\widetilde{g}_{21}(s)\}\widetilde{m}_1(s) \\ &= H_1(s)[(p+qs)A(\alpha,s)\widetilde{g}_{22}(s) - s] - H_2(s)(p+qs)A(\alpha,s)\widetilde{g}_{12}(s). \end{aligned} \tag{2.4}$$

For notational convenience, we define

$$\begin{aligned} &h_i(0) = \alpha pe_i,\ \overline{g}_{ij}(0) = \alpha pg_{ij}(0), \quad \overline{g}_{ij}(1) = \alpha pg_{ij}(1) + (1-\alpha)pg_{ij}(0) + \alpha qg_{ij}(0), \quad i \neq j, \\ &\overline{g}_{ii}(1) = \alpha pg_{ii}(1) + (1-\alpha)pg_{ii}(0) + \alpha qg_{ii}(0) - 1, \\ &\overline{g}_{ij}(k) = \alpha pg_{ij}(k) + \alpha qg_{ij}(k-1) + (1-\alpha)pg_{ij}(k-1) + (1-\alpha)qg_{ij}(k-2),\ k \in \mathbb{N}\setminus\{0,1\}, \\ &h_i(k) = -\alpha p\xi_i(k-1) - (1-\alpha)(p\eta_i(k-1) + N_i(k-1)) - \alpha q\eta_i(k-1) - (1-\alpha)q\zeta_i(k-1), \\ &k = 1,2,\cdots,x, \\ &h_i(k) = -\alpha p\xi_i(k-1) - (1-\alpha)(p\eta_i(k-1) - \alpha q\eta_i(k-1) - (1-\alpha)q\zeta_i(k-1), \\ &k = x+1, x+2, \cdots, \\ &f_k = \sum_{n=0}^{k}[\overline{g}_{11}(n)\overline{g}_{22}(k-n) - \overline{g}_{21}(n)\overline{g}_{12}(k-n)], \\ &g_k^{(1)} = \sum_{n=0}^{k} m_1(n)f_{k-n}, \quad A_k^{(1)} = \sum_{n=0}^{k}[h_1(n)\overline{g}_{22}(k-n) - h_2(n)\overline{g}_{12}(k-n)], \quad k \in \mathbb{N}. \end{aligned}$$

Let $\widetilde{g}^{(1)}(s)$, $\widetilde{f}(s)$, and $\widetilde{A}^{(1)}(s)$ be the generating functions of $g_k^{(1)}$, f_k, and $A_k^{(1)}$ respectively. From (2.4) we get

$$\widetilde{g}^{(1)}(s) = \widetilde{f}(s)\widetilde{m}_1(s) = \widetilde{A}^{(1)}(s). \tag{2.5}$$

Inverting the equation (2.5) gives

$$\sum_{n=0}^{k} m_1(n)f_{k-n} = A_k^{(1)}, \quad k \in \mathbb{N}. \tag{2.6}$$

Similarly, we have

$$\sum_{n=0}^{k} m_2(n)f_{k-n} = A_k^{(2)}, \quad k \in \mathbb{N}. \tag{2.7}$$

Theorem 1. For $i = 1, 2$ and $k \in \mathbb{N}^+$, the ruin probability satisfies the recursive formula as follows

$$m_i(k) = \begin{cases} \frac{1}{f_0}[A_k^{(i)} - \sum_{n=0}^{k-1} m_i(n)f_{k-n}], & \text{if } f_0 \neq 0, \\ \frac{1}{f_1}[A_{k+1}^{(i)} - \sum_{n=0}^{k-1} m_i(n)f_{k+1-n}], & \text{if } f_0 = 0,\ f_1 \neq 0. \end{cases} \tag{2.8}$$

Proof. From (2.6) and (2.7), we only need to show that the safty load condition will not hold for $f_0 = f_1 = 0$. Note that

$$\begin{aligned} f_1 &= \alpha pg_{11}(0)[\alpha pg_{22}(1) + (1-\alpha)pg_{22}(0) + \alpha qg_{22}(0) - 1] - \alpha pg_{21}(0)[\alpha pg_{12}(1) + (1-\alpha)pg_{12}(0) \\ &+ \alpha qg_{12}(0)] + \alpha pg_{22}(0)[\alpha pg_{11}(1) + (1-\alpha)pg_{11}(0) + \alpha qg_{11}(0) - 1] - \alpha pg_{12}(0)[\alpha pg_{21}(1) \\ &+ (1-\alpha)pg_{21}(0) + \alpha qg_{21}(0)] \leq 0. \end{aligned}$$

If $f_1 = 0$, then $g_{11}(0) = g_{22}(0) = 0$ and

$$\alpha p g_{21}(0)[\alpha p g_{12}(1) + (1-\alpha) p g_{12}(0) + \alpha q g_{12}(0)] = 0, \tag{2.9}$$

$$\alpha p g_{12}(0)[\alpha p g_{21}(1) + (1-\alpha) p g_{21}(0) + \alpha q g_{21}(0)] = 0. \tag{2.10}$$

From $f_0 = \alpha^2 p^2 (g_{11}(0) g_{22}(0) - g_{21}(0) g_{12}(0)) = 0$, we know $g_{21}(0) = g_{12}(0) = 0$. Consider two situations

(i) $g_{12}(0) = 0$ but $g_{21}(0) \neq 0$. From (2.9), we know $g_{12}(1) = 0$, and get $\mu_1 \geq 1 + p_{12}$, $\mu_2 \geq 1 - g_{21}(0)$, but the only initial distribution is $\pi = (\pi_1, \pi_2) = (\frac{p_{21}}{p_{21}+p_{12}}, \frac{p_{12}}{p_{21}+p_{12}})$. Therefore,

$$\pi_1\mu_1 + \pi_2\mu_2 \geq \frac{p_{21}}{p_{21}+p_{12}}(1+p_{12}) + \frac{p_{12}}{p_{21}+p_{12}}(1 - g_{21}(0)) \geq 1.$$

(ii) $g_{21}(0) = 0$ but $g_{12}(0) \neq 0$. From (2.10), we know $g_{21}(1) = 0$. We have $\mu_1 \geq 1 - g_{12}(0)$, but $\mu_2 \geq 2p_{21} + p_{22} = 1 + p_{21}$. So we have

$$\pi_1\mu_1 + \pi_2\mu_2 \geq \frac{p_{21}(1 - g_{12}(0)) + p_{12}(1+p_{21})}{p_{21}+p_{12}} \geq 1.$$

This complete the proof of theorem 1. □

To apply the recursive formula (2.8), we need $2(x+1)$ equations to solve the problem of the values of $m_i(0), m_i(1), \cdots, m_i(x), i = 1, 2$. Obviously, let $u = 0, 1, \cdots, x-1$ in (2.1), we can get $2x$ equations. Now we aim to find another two equations. In what follows, we borrow the same symbols as Yuen et al. [5]. For $u, z \in \mathbb{N}$, $y \in \mathbb{N}^+$ and $i, j = 1, 2$, define

$$f_{ij}(u,z,y) = \mathbb{P}(\tau < \infty, J_\tau = j, U_{\tau^-} = z, |U_\tau| = y | U_0 = u, J_0 = i),$$

$$m_{ij}(u) = E(\omega(U_{\tau^-}, |U_\tau|) 1_{(\tau<\infty, J_\tau = j)} | U_0 = u, J_0 = i).$$

From the same method in [5], we have

$$m_i(x) = \sum_{k=1}^{2}\sum_{z=0}^{\infty}\sum_{y=1}^{x} f_{ik}(0,z,y) m_k(x-y) + \sum_{z=0}^{\infty}\sum_{y=x+1}^{\infty} f_i(0,z,y)\omega(x+z, y-x), i,j = 1,2, \tag{2.11}$$

where $f_i(0,z,y) = \sum_{j=1}^{2} f_{ij}(0,z,y)$. Since $m_{ij}(0) = f_{ij}(0,z,y)$, we should give the values of $m_{ij}(0)$ under the condition of $x = 0$.

2.1 The first equation between $m_1(0)$ and $m_2(0)$

Now we assume $x = 0$ and denote by

$$\xi_{il}(u) = \sum_{k=u+2}^{\infty} g_{il}(k)\omega(u+1, k-u-1), \quad \eta_{il}(u) = \sum_{k=u+1}^{\infty} g_{il}(k)\omega(u, k-u),$$

$$\zeta_{il}(u) = \sum_{k=u}^{\infty} g_{il}(k)\omega(u-1, k-u+1), \quad i, l = 1, 2,$$

then equation (2.2) can be replaced by

$$m_{il}(u) = \alpha\{p[\sum_{j=1}^{2}\sum_{k=0}^{u+1} g_{ij}(k) m_{jl}(u+1-k) + \xi_{il}(u)] + q[\sum_{j=1}^{2}\sum_{k=0}^{u} g_{ij}(k) m_{jl}(u-k) + \eta_{il}(u)]\}$$
$$+ (1-\alpha)\{p[\sum_{j=1}^{2}\sum_{k=0}^{u} g_{ij}(k) m_{jl}(u-k) + \eta_{il}(u)] + q[\sum_{j=1}^{2}\sum_{k=0}^{u-1} g_{ij}(k) m_{jl}(u-1-k) + \zeta_{il}(u)]\}.$$

It is easy to see that

$$\lim_{s\to 1}\widetilde{f}(s)=\lim_{s\to 1}\{[(p+qs)A(\alpha,s)\widetilde{g}_{11}(s)-s][(p+qs)A(\alpha,s)\widetilde{g}_{22}(s)-s] \\ -(p+qs)^2A^2(\alpha,s)\widetilde{g}_{12}(s)\widetilde{g}_{21}(s)\}=0.$$

Therefore, if $\lim_{s\to 1}\widetilde{m}_1(u)=\sum_{u=0}^{\infty}m_1(u)<\infty$, from (2.5) we get

$$\lim_{s\to 1}\widetilde{A}^{(1)}(s)=-p_{21}[\alpha pe_1-\alpha p\sum_{u=0}^{\infty}\xi_1(u)-(1-\alpha)p\sum_{u=0}^{\infty}\eta_1(u)-\alpha q\sum_{u=0}^{\infty}\eta_1(u)-(1-\alpha)q\sum_{u=1}^{\infty}\zeta_1(u)] \\ -p_{12}[\alpha pe_2-\alpha p\sum_{u=0}^{\infty}\xi_2(u)-(1-\alpha)p\sum_{u=0}^{\infty}\eta_2(u)-\alpha q\sum_{u=0}^{\infty}\eta_2(u)-(1-\alpha)q\sum_{u=1}^{\infty}\zeta_2(u)]=0,$$

which is equivalent to

$$\begin{aligned}&pm_1(0)(g_{11}(0)p_{21}+g_{21}(0)p_{12})+pm_2(0)(g_{12}(0)p_{21}+g_{22}(0)p_{12})\\&=pp_{21}\sum_{u=0}^{\infty}\xi_1(u)+pp_{12}\sum_{u=0}^{\infty}\xi_2(u)+qp_{21}\sum_{u=0}^{\infty}\eta_1(u)+qp_{12}\sum_{u=0}^{\infty}\eta_2(u)\\&+\frac{(1-\alpha)}{\alpha}(pp_{21}\sum_{u=0}^{\infty}\eta_1(u)+pp_{12}\sum_{u=0}^{\infty}\eta_2(u)+qp_{21}\sum_{u=1}^{\infty}\zeta_1(u)+qp_{12}\sum_{u=1}^{\infty}\zeta_2(u)).\end{aligned}\tag{2.12}$$

Now we check that (2.12) still holds in the case of $\sum_{u=0}^{\infty}m_1(u)=\infty$. Let $\widetilde{\phi}_i(s)$ be the generating function of $\phi_i(u)$ for $i=1,2$. By imitating the proof of Proposition 2 in [5], one can easily obtain that

$$\lim_{s\to 1}\widetilde{f}(s)\widetilde{\phi}_i(s)=-\widetilde{f}'(1)<\infty.\tag{2.13}$$

Let $\widetilde{\psi}_1(s)$ represent the generating function of $\psi_1(u)$. Then equation (2.13) leads to

$$\lim_{s\to 1}\widetilde{f}(s)\widetilde{\psi}_1(s)=\lim_{s\to 1}\widetilde{f}(s)(\frac{1}{1-s}-\widetilde{\phi}_1(s))=\lim_{s\to 1}\frac{\widetilde{f}(s)}{1-s}-\lim_{s\to 1}\widetilde{f}(s)\widetilde{\phi}_1(s)=\widetilde{f}'(1)-\widetilde{f}'(1)=0,$$

which further implies that

$$\lim_{s\to 1}\widetilde{f}(s)\widetilde{m}_1(s)=\lim_{s\to 1}\widetilde{f}(s)\widetilde{\psi}_1(s)=0.$$

From (2.5), we obtain $\lim_{s\to 1}\widetilde{A}^{(1)}(s)=0$, and this shows that equation (2.12) is still true.

2.2 The second equation between $m_1(0)$ and $m_2(0)$

Note that $f_0=\alpha^2p^2[g_{11}(0)g_{22}(0)-g_{12}(0)g_{21}(0)]$. We first consider the spcial case of $f_0=0$. Denote by

$$K_1=\alpha^2p^2[g_{11}(1)g_{22}(0)-g_{12}(0)g_{21}(1)]-\alpha pg_{22}(0)\le 0,$$
$$K_2=\alpha^2p^2[g_{22}(0)g_{12}(1)-g_{12}(0)g_{22}(1)]+\alpha pg_{12}(0)\ge 0,$$

in terms of equation (2.6), we have

$$\begin{aligned}&K_1m_1(0)+K_2m_2(0)=\alpha^2p^2(g_{12}(0)\xi_2(0)-g_{22}(0)\xi_1(0))+(\alpha(1-\alpha)p^2\\&+\alpha^2pq)(g_{12}(0)\eta_2(0)-g_{22}(0)\eta_1(0))+\alpha(1-\alpha)pq(g_{12}(0)\zeta_2(0)-g_{22}(0)\zeta_1(0)).\end{aligned}\tag{2.14}$$

Since $K_1=K_2=0$ holds if only if $g_{12}(0)=g_{22}(0)=0$, and this gives $e_1=g_{11}(0)m_1(0), e_2=g_{21}(0)m_1(0)$. Therefore, we have

$$\begin{aligned}f_1m_2(0)=A_1^{(2)}&=[\alpha^2p^2(g_{21}(0)g_{11}(1)-g_{11}(0)g_{21}(1))-\alpha pg_{21}(0)]m_1(0)\\&+\alpha^2p^2(g_{21}(0)\xi_1(0)-g_{11}(0)\xi_2(0))+\alpha(1-\alpha)p^2(g_{21}(0)\eta_1(0)\\&-g_{11}(0)\eta_2(0))+\alpha^2pq(g_{21}(0)\eta_1(0)-g_{11}(0)\eta_2(0)).\end{aligned}\tag{2.15}$$

Seondly, we consider the case of $f_0 > 0$. Some rearrangements leads to

$$\begin{aligned}\widetilde{f}'(s) &= \{qA(\alpha,s)\widetilde{g}_{11}(s) + (p+qs)[(1-\alpha)\widetilde{g}_{11}(s)A(\alpha,s)\widetilde{g}'_{11}(s)] - 1\}[(p+qs)A(\alpha,s)\widetilde{g}_{22}(s) - s] \\ &+ [(p+qs)A(\alpha,s)\widetilde{g}_{11}(s) - s]\{qA(\alpha,s)\widetilde{g}_{22}(s) + (p+qs)[(1-\alpha)\widetilde{g}_{22}(s) + A(\alpha,s)\widetilde{g}'_{22}(s)] - 1\} \\ &- 2q(p+qs)A^2(\alpha,s)\widetilde{g}_{12}(s)\widetilde{g}_{21}(s) - (p+qs)^2[2(1-\alpha)A(\alpha,s)\widetilde{g}_{12}(s)\widetilde{g}_{21}(s) \\ &+ A^2(\alpha,s)(\widetilde{g}'_{12}(s)\widetilde{g}_{21}(s) + \widetilde{g}_{12}(s)\widetilde{g}'_{21}(s))],\end{aligned}$$

which means that

$$\begin{aligned}\widetilde{f}'(1) &= -p_{21}[qp_{11} + (1-\alpha)p_{11} + \mu_{11} - 1] - p_{12}[qp_{22} + (1-\alpha)p_{22} + \mu_{22} - 1] \\ &- 2qp_{12}p_{21} - 2(1-\alpha)p_{12}p_{21} - p_{21}\mu_{12} - p_{12}\mu_{21} \\ &= \alpha(p_{12}+p_{21}) - (p_{12}\mu_1 + p_{21}\mu_2) - q(p_{12}+p_{21}) > 0.\end{aligned}$$

Note that $\widetilde{f}(1) = 0$, there exists a $\rho \in (0,1)$ such that $\widetilde{f}(\rho) = 0$, i.e., $\widetilde{A}^{(1)}(\rho) = 0$. Equivalently, we have

$$\begin{aligned}& p[(p+q\rho)A(\alpha,\rho)(g_{11}(0)\widetilde{g}_{22}(\rho) - g_{21}(0)\widetilde{g}_{12}(\rho)) - \rho g_{11}(0)]m_1(0) \\ &+ p[(p+q\rho)A(\alpha,\rho)(g_{12}(0)\widetilde{g}_{22}(\rho) - g_{22}(0)\widetilde{g}_{12}(\rho)) - \rho g_{12}(0)]m_2(0) \\ &= \rho p\{[\widetilde{\xi}_1(\rho) + \frac{1-\alpha}{\alpha}\widetilde{\eta}_1(\rho)][(p+q\rho)A(\alpha,\rho)\widetilde{g}_{22}(\rho) - \rho] \\ &- [\widetilde{\xi}_2(\rho) - \frac{1-\alpha}{\alpha}\widetilde{\eta}_2(\rho)](p+q\rho)A(\alpha,\rho)\widetilde{g}_{12}(\rho)\} \\ &+ \rho q\{[\widetilde{\eta}_1(\rho) + \frac{1-\alpha}{\alpha}\widetilde{\zeta}_1(\rho)][(p+q\rho)A(\alpha,\rho)\widetilde{g}_{22}(\rho) - \rho] \\ &- [\widetilde{\eta}_2(\rho) + \frac{1-\alpha}{\alpha}\widetilde{\zeta}_2(\rho)](p+q\rho)A(\alpha,\rho)\widetilde{g}_{12}(\rho)\}.\end{aligned} \tag{2.16}$$

Finally, consider the case of $f_0 < 0$, then $\widetilde{f}(0) = f_0 = 0$. On the other hand,

$$\begin{aligned}\widetilde{f}(-1) &= [(2p-1)(2\alpha-1)\widetilde{g}_{11}(-1) + 1][(2p-1)(2\alpha-1)\widetilde{g}_{22}(-1) + 1] \\ &- (2p-1)^2(2\alpha-1)^2\widetilde{g}_{21}(-1)\widetilde{g}_{12}(-1) > (1-\widetilde{g}_{11}(1))(1-\widetilde{g}_{22}(1)) - \widetilde{g}_{21}(1)\widetilde{g}_{12}(1) \\ &= (1-p_{11})(1-p_{22}) - p_{21}p_{12} = 0, \quad \forall\alpha \in (0,1], \quad \forall p \in (0,1],\end{aligned}$$

which implies that $\widetilde{f}(\rho) = 0$ holds for some $\rho \in (-1,0)$, that is, $\widetilde{A}^{(1)}(\rho) = 0$. Then (2.16) still holds.

3 Numerical Analysis

We remark that the risk model in [5] can be covered by letting $p = 1$ in the model proposed in the present paper. To give a comparision, we cite the numerical example in [5]. The distribution of $g_{ij}(k)$ is as follows

Table 1. The distribution of claims.

k	$g_{11}(k)$	$g_{12}(k)$	$g_{21}(k)$	$g_{22}(k)$
0	5/8	0	0	0
1	1/8	1/8	0	1/6
2	1/8	0	1/2	1/6
3	0	0	1/6	0
≥ 4	0	0	0	0

Let $\alpha = 0.8, 0.85, 0.9$, $x = 0$, $p = 0.95$, $q = 0.05$. Direct calculation gives

$$p_{12} = \frac{1}{8}, \quad p_{21} = \frac{2}{3}, \quad \mu_1 = \frac{1}{2}, \quad \mu_2 = 2.$$

Note that $f_0 = 0$, $g_{12}(0) = g_{22}(0) = 0$. According to (2.12) and (2.15), the values of $\{f_{ij}(0, z, y), i, j = 1, 2\}$ can be obtained with $\mu_1 = \frac{1}{2}$ and $x = 0$.

$$f_{1j}(0, z, y) = \frac{12}{5}p[\frac{2}{3}g_{1j}(z + y) + \frac{1}{8}g_{2j}(z + y)](1_{z\geq 1} + \frac{1-\alpha}{\alpha}) + \frac{24}{5}q[\frac{2}{3}g_{1j}(z + y) + \frac{1}{8}g_{2j}(z + y)],$$

$$f_{2j}(0, z, y) = \frac{6\alpha p}{6 - \alpha p}g_{2j}(z + y)1_{(z=1)} + \frac{6(1-\alpha)p}{6 - \alpha p}g_{2j}(z + y)1_{(z=0)} + \frac{6\alpha q}{6 - \alpha p}g_{2j}(z + y)1_{(z=0)}.$$

Table 2 to Table 5 are the values of $f_{ij}(0, z, y)$, $i, j = 1, 2$ and Table 6 is the value of $\psi_i(u)$.

Table 2. The values of $f_{11}(0, z, y)$.

$f_{11}(0, z, y)$	$y = 0$	$y = 1$	$y = 2$	$y = 3$	$y \geq 4$
$z = 0$	$\frac{(1-\alpha)p}{\alpha} + 2q$	$\frac{(1-\alpha)p}{5\alpha} + \frac{2}{5}q$	$\frac{7(1-\alpha)p}{20\alpha} + \frac{7}{10}q$	$\frac{(1-\alpha)p}{20\alpha} + \frac{1}{10}q$	0
$z = 1$	$\frac{p}{5\alpha} + \frac{2}{5}q$	$\frac{7p}{20\alpha} + \frac{7}{10}q$	$\frac{p}{20\alpha} + \frac{1}{10}q$	0	0
$z = 2$	$\frac{7p}{20\alpha} + \frac{7}{10}q$	$\frac{p}{20\alpha} + \frac{1}{10}q$	0	0	0
$z = 3$	$\frac{p}{20\alpha} + \frac{1}{10}q$	0	0	0	0
$z \geq 4$	0	0	0	0	0

Table 3. The values of $f_{12}(0, z, y)$.

$f_{12}(0, z, y)$	$y = 0$	$y = 1$	$y = 2$	$y \geq 3$
$z = 0$	0	$\frac{(1-\alpha)p}{4\alpha} + \frac{1}{2}q$	$\frac{(1-\alpha)p}{20\alpha} + \frac{1}{10}q$	0
$z = 1$	$\frac{p}{4\alpha} + \frac{1}{2}q$	$\frac{p}{20\alpha} + \frac{1}{10}q$	0	0
$z = 2$	$\frac{p}{20\alpha} + \frac{1}{10}q$	0	0	0
$z \geq 3$	0	0	0	0

Table 4. The values of $f_{21}(0, z, y)$.

$f_{21}(0, z, y)$	$y = 0$	$y = 1$	$y = 2$	$y = 3$	$y \geq 4$
$z = 0$	0	0	$\frac{3(1-\alpha)p+3\alpha q}{6-\alpha p}$	$\frac{(1-\alpha)p+\alpha q}{6-\alpha p}$	0
$z = 1$	0	$\frac{3\alpha p}{6-\alpha p}$	$\frac{\alpha p}{6-\alpha p}$	0	0
$z \geq 2$	0	0	0	0	

Table 5. The values of $f_{22}(0, z, y)$.

$f_{22}(0, z, y)$	$y = 0$	$y = 1$	$y = 2$	$y \geq 3$
$z = 0$	0	$\frac{(1-\alpha)p+6\alpha q}{6-\alpha p}$	$\frac{(1-\alpha)p+6\alpha q}{6-\alpha p}$	0
$z = 1$	$\frac{\alpha p}{6-\alpha p}$	$\frac{\alpha p}{6-\alpha p}$	0	0
$z \geq 2$	0	0	0	0

Table 6 describes the behavior of $\psi_i(u), i = 1, 2$ with respect to initial surplus u for fixed p. As is expected, $\psi_i(u)$ decreases as the initial surplus u increases. On the other hand, for fixed u, $\psi_i(u)$ always decreases as the value of α increases. Compare with Table 6 in [5], we see that the corresponding values of $\psi_i(u)$ become larger for the same initial u. This is, heuristically, due to the fact that the insurer receives one unit of premiums with probability $p < 1$. This illustrates the fact that the randomness of premiums has a significant effect on the ruin probabilities.

Table 6. The values of $\psi_i(u)(p = 0.95, q = 0.05)$.

		$\psi_1(u)$			$\psi_2(u)$	
u	$\alpha = 0.8$	$\alpha = 0.85$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.85$	$\alpha = 0.9$
0	0.9475	0.8497	0.7628	1	1	1
1	0.9392	0.7833	0.6529	0.9955	0.9433	0.8905
2	0.9004	0.6989	0.5435	0.9644	0.8477	0.7398
3	0.8513	0.6133	0.4449	0.9267	0.7533	0.6067
4	0.7847	0.5213	0.3514	0.8581	0.6390	0.4696
5	0.7033	0.4263	0.2649	0.7710	0.5185	0.3409
6	0.6083	0.3301	0.1860	0.6697	0.3975	0.2249
7	0.5018	0.2348	0.1150	0.5555	0.2779	0.1216
8	0.3855	0.1421	0.0523	0.4306	0.1618	0.0311
9	0.2618	0.0535	0	0.2974	0.0513	0
10	0.1330	0	0	0.1584	0	0
11	0.0015	0	0	0.0162	0	0

4 Conclusions

In this paper,we mainly investigate the discrete semi-Markov risk model with dividends and stochastic premiums. By using the technique of probability generating function, recursive equations for the expected penalty function are derived.

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Competing Interests

Author has declared that no competing interests exist.

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