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Hankel Determinant for Certain Subclasses of Analytic Function

Hamzat Jamiu Olusegun[1*] and Oni Abiola Ayobami[1]

[1]Department of Pure and Applied Mathematics, Ladoke Akintola University of Technology, P.M.B. 4000, Ogbomoso, Nigeria.

Authors' contributions

This work was carried out in collaboration between both authors. Author HJO designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author OAA managed the analyses of the study. Authors HJO and OAA managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

Let S denote the class of analytic functions normalized and univalent in the open unit disk. $U = \{z : |z| < 1\}$. The prime focus of the present paper is to obtain sharp upper bounds for the functional $|a_{\alpha+2} - \mu a_{\alpha+1}^2|$ and $|a_{\alpha+1}a_{\alpha+3} - a_{\alpha+2}^2|$ for functions belonging to the class $T_n^{\alpha}(a, c, \beta, \lambda, l)$.

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**Corresponding author: E-mail: emmanther2012@gmail.com;*

1 Introduction

Let *A* denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

in the unit disk $U = \{z : |z| < 1\}$ and normalized with $f(0) = f'(0) - 1 = 0$. Also, we denote the subclass of *A* consisting of univalent functions $f(z)$ in the unit disk *U* by *S*. Now, the following classes of analytic functions shall be recalled.

Let $f \in A$. Then, $f \in S^*$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad z \in U. \tag{2}$$

This is called the class of starlike functions. Also, let $f \in A$. Then $f \in C$ if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \qquad z \in U. \tag{3}$$

This is referred to as the class of convex functions. These two classes have been widely studied by various authors such as [1-4] to mention but few (see also [5]). The theory of analytic functions has wide application in many physical problems such as fluid flows, heat conduction, aerodynamics and so on.

Now, let $A(\alpha)$ denote the class of functions of the form:

$$f_\alpha(z) = z^\alpha + \sum_{k=\alpha+1}^{\infty} a_k z^k, \qquad \alpha \in N = \{1, 2, 3, \ldots\} \tag{4}$$

which are analytic and α-valent in the open unit disk *U*. Suppose that we define the α-modified Catas derivative operator $I_\alpha^n(\lambda, l)$ as follows:

$$I_\alpha^0(\lambda, l) f(z) = f(z) = z^\alpha + \sum_{k=\alpha+1}^{\infty} a_k z^k \tag{5}$$

$$\begin{aligned} I_\alpha^1(\lambda, l) f(z) &= I_\alpha^0(\lambda, l) f(z)\left(\frac{1-\lambda+l}{1+l}\right) + \left(I_\alpha^0(\lambda, l) f(z)\right)'\left(\frac{\lambda z}{1+l}\right) \\ &= \left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right) z^\alpha + \sum_{k=\alpha+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right) a_k z^k \end{aligned} \tag{6}$$

$$I_{\alpha}^{2}(\lambda,l)f(z)=I_{\alpha}^{1}(\lambda,l)f(z)\left(\frac{1-\lambda+l}{1+l}\right)+\left(I_{\alpha}^{1}(\lambda,l)f(z)\right)'\left(\frac{\lambda z}{1+l}\right)$$
$$=\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^{2}z^{\alpha}+\sum_{k=\alpha+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{2}a_k z^k \quad (7)$$

$$I_{\alpha}^{3}(\lambda,l)f(z)=I_{\alpha}^{2}(\lambda,l)f(z)\left(\frac{1-\lambda+l}{1+l}\right)+\left(I_{\alpha}^{2}(\lambda,l)f(z)\right)'\left(\frac{\lambda z}{1+l}\right)$$
$$=\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^{3}z^{\alpha}+\sum_{k=\alpha+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{3}a_k z^k \quad (8)$$

and in general

$$I_{\alpha}^{n}(\lambda,l)f(z)=I_{\alpha}^{n-1}(\lambda,l)f(z)\left(\frac{1-\lambda+l}{1+l}\right)+\left(I_{\alpha}^{n-1}(\lambda,l)f(z)\right)'\left(\frac{\lambda z}{1+l}\right)$$
$$=\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^{n}z^{\alpha}+\sum_{k=\alpha+1}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}a_k z^k \quad (9)$$

$\lambda\ge 0;\ \ l\ge 0;\ \ n\in N_0;\ \alpha\in N\ \ and\ \ z\in U$.

By specializing the parameters involved in (9), various derivative operators (known and new) are obtained (see [6-10] among others).

Let us also define the function $\varphi^{\alpha}(a,c;z)$ by

$$\varphi^{\alpha}(a,c;z)=z^{\alpha}+\sum_{k=\alpha+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k \quad (10)$$

$\alpha\in N;\ a\in\Re;\ c\in\Re-\{0,-1,-2,...\}\ and\ \ z\in U$, where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k=\frac{\gamma(a+k)}{\gamma(a)}=\begin{cases}1, & k=0\\ a(a+1)(a+2)...(a+k-1), & k\in N.\end{cases} \quad (11)$$

Or

$$(a)_k=\frac{\gamma(a+k)}{\gamma(a)}=\frac{1}{a+k}\left\{\prod_{j=1}^{k+1}(a+j-1)\right\},\qquad k\in N_0=N\cup\{0\}\ . \quad (12)$$

With the aid of (9) and (10), we define a linear operator

$$J_n^{\alpha}(a,c)f(z)=\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^{n}z^{\alpha}+\sum_{k=\alpha+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{n}a_k z^k \quad (13)$$

Remark

For $\lambda = 1$, $n = l = 0$ and $\alpha = 1$ operator (13) cracks down to Carlson-Shaffer operator(see [11]), and also for different values of the parameters involved the operator in (13) reduces to the likes of the Saitoh operator [12] and recently the Mahzoon-Lotha [13] operator. In particular, if $\lambda = 1$, $\alpha = 1$ and $a = c$ we have the celebrated Salagean derivative operator. However, our prime focus is to examine the coefficient bound, Fekete-Szego problem and the second Hankel determinant for functions in the analytic class $\mathrm{T}_n^{\alpha}(a, c, \beta, \lambda, l)$.

With this in mind, we define the following class of analytic functions.

Suppose that the function $f_{\alpha}(z)$ is of the form (4). Then $f_{\alpha}(z) \in T_n^{\alpha}(a, c, \beta, \lambda, l)$ if it satisfies the condition that:

$$\Re\left\{ \frac{z\left[J_n^{\alpha}(a,c) f_{\alpha}(z) + \beta z\left(J_n^{\alpha}(a,c) f_{\alpha}(z)\right)'\right]'}{\alpha\left[J_n^{\alpha}(a,c) f_{\alpha}(z) + \beta z\left(J_n^{\alpha}(a,c) f_{\alpha}(z)\right)'\right]} \right\} > 0, \tag{14}$$

$$\alpha \in N;\ \ a \in \Re;\ \ c \in \Re - \{0, -1, -2, \ldots\},\ \ 0 \le \beta < 1 \ and\ \ z \in U.$$

Noonan and Thomas [14] stated the q^{th} Hankel determinant for $q \ge 1$ and $n \ge 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}$$

This determinant has been repeatedly investigated by several authors and researchers (see [15-17, 1, 18-20]).

In particular, one can observe that the Fekete and Szego functional is $H_2(1)$. Fekete and Szego [1] then further generalized the estimate $\left|a_3 - \mu a_2^2\right|$ where μ is real and $f \in S$.

Also, for the purpose of the present investigation we consider the second Hankel determinant in which $q = 2$ and $n = 2$, such that

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \qquad \alpha \in N.$$

2 Preliminary Results

Let P (Class of Caratheodory functions) be the family of all functions *p* analytic in *U* for which $p(0) = 1$, $\Re\{p(z)\} > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \qquad z \in D \tag{15}$$

in the unit disk *D* (see [21]).

Lemma 2.1 [21]. If $p \in \mathrm{P}$ (class of Caratheodory functions), then $|p_k| \leq 2$ for each $k, (k = 1, 2, \ldots)$.

Lemma 2.2 [18]. If $p \in \mathrm{P}$, then

$$p_2 = \frac{1}{2}\left[p_1^2 + (4 - p_1^2)x\right],$$

And

$$p_3 = \frac{1}{4}\left[p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z\right]$$

for some value of x, z such that $|x| \leq 1$ *and* $|z| \leq 1$.

3 Main Results

Theorem 3.1. Let $f_\alpha(z) \in T_n^\alpha(a, c, \beta, \lambda, l)$ Then, for $\lambda \geq 0;\ l \geq 0;\ a \in \Re;\ c \in \Re - \{0, -1, -2, \ldots\};$ $\alpha \in N; 0 \leq \beta < 1;\ n \in N_0$ *and* $z \in U$

$$|a_{\alpha+1}| \leq \frac{(c)_\alpha}{(a)_\alpha} \cdot \frac{p_1}{\psi\, \gamma_1\, \phi_1}, \quad |a_{\alpha+2}| \leq \frac{(c)_{\alpha+1}}{(a)_{\alpha+1}} \cdot \frac{(\alpha p_1^2 + p_2)}{2\psi\, \gamma_2\, \phi_2}, \quad |a_{\alpha+3}| \leq \frac{(c)_{\alpha+2}}{(a)_{\alpha+2}} \cdot \frac{(\alpha^2\, p_1^3 + 3\alpha\, p_1\, p_2 + 2p_3)}{6\psi\, \gamma_3\, \phi_3}$$

where

$$\gamma_1 = [1 + \beta(1 + \alpha)], \quad \gamma_2 = [1 + \beta(2 + \alpha)], \quad \gamma_3 = [1 + \beta(3 + \alpha)], \phi_1 = \left(\frac{1 + \alpha\lambda + l}{1 + \lambda(\alpha - 1) + l}\right)^n,$$

$$\phi_2 = \left(\frac{1 + \lambda(1 + \alpha) + l}{1 + \lambda(\alpha - 1) + l}\right)^n, \quad \phi_3 = \left(\frac{1 + \lambda(2 + \alpha) + l}{1 + \lambda(\alpha - 1) + l}\right)^n \quad and \quad \psi = \frac{1}{\alpha(1 + \alpha\beta)}$$

Proof: Supposing $f_\alpha(z) \in T_n^\alpha(a, c, \beta, \lambda, l)$, then we have that

$$\frac{z\left[J_n^\alpha(a,c)\,f(z) + \beta\, z\,(J_n^\alpha(a,c)\,f(z))'\right]'}{\alpha\left[J_n^\alpha(a,c)\,f(z) + \beta\, z\,(J_n^\alpha(a,c)\,f(z))'\right]} = p(z). \tag{16}$$

where p(z) is as defined in (15). This implies that

$$1 + \sum_{k=\alpha+1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} k\ (1 + \beta k)\psi \left(\frac{1 + \lambda(k-1) + l}{1 + \lambda(\alpha - 1) + l}\right)^n a_k\, z^{k-\alpha}$$

$$= \left(1+p_1z+p_2z^2+p_3z^3+p_4z^4+...\right)\left(1+\sum_{k=\alpha+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}\alpha(1+Bk)\psi\left(\frac{1+\lambda(k-1)+l}{1+\lambda(\alpha-1)+l}\right)^n a_k z^{k-\alpha}\right) \quad (17)$$

By comparing the coefficients of the like power of *z* in (17), we obtain

$$\frac{(a)_\alpha}{(c)_\alpha}(1+\alpha)[1+\beta(1+\alpha)]\psi\left(\frac{1+\alpha\lambda+l}{1+\lambda(\alpha-1)+l}\right)^n a_{\alpha+1}$$
$$=\frac{(a)_\alpha}{(c)_\alpha}\alpha\,[1+\beta(1+\alpha)]\psi\left(\frac{1+\alpha\lambda+l}{1+\lambda(\alpha-1)+l}\right)^n a_{\alpha+1}+p_1, \quad (18)$$

$$\frac{(a)_{\alpha+1}}{(c)_{\alpha+1}}(2+\alpha)[1+\beta(2+\alpha)]\psi\left(\frac{1+\lambda(1+\alpha)+l}{1+\lambda(\alpha-1)+l}\right)^n a_{\alpha+2}$$
$$=\frac{(a)_{\alpha+1}}{(c)_{\alpha+1}}\alpha[1+\beta(2+\alpha)]\psi\left(\frac{1+\lambda(1+\alpha)+l}{1+\lambda(\alpha-1)+l}\right)^n a_{\alpha+2}$$
$$+\frac{(a)_\alpha}{(c)_\alpha}\alpha\,[1+\beta(1+\alpha)]\psi\left(\frac{1+\alpha\lambda+l}{1+\lambda(\alpha-1)+l}\right)^n a_{\alpha+1}+p_2 \quad (19)$$

and

$$\frac{(a)_{\alpha+2}}{(c)_{\alpha+2}}(3+\alpha)[1+\beta(3+\alpha)]\psi\left(\frac{1+\lambda(2+\alpha)+l}{1+\lambda(\alpha-1)+l}\right)^n a_{\alpha+3}$$
$$=\frac{(a)_{\alpha+2}}{(c)_{\alpha+2}}\alpha[1+\beta(3+\alpha)]\psi\left(\frac{1+\lambda(2+\alpha)+l}{1+\lambda(\alpha-1)+l}\right)^n a_{\alpha+3}+\frac{(a)_{\alpha+1}}{(c)_{\alpha+1}}\alpha[1+\beta(2+\alpha)]\psi\left(\frac{1+\lambda(1+\alpha)+l}{1+\lambda(\alpha-1)+l}\right)^n a_{\alpha+2}p_1$$
$$+\frac{(a)_\alpha}{(c)_\alpha}\alpha\,[1+\beta(1+\alpha)]\psi\left(\frac{1+\alpha\lambda+l}{1+\lambda(\alpha-1)+l}\right)^n a_{\alpha+1}p_2+p_3 \quad (20)$$

and this ends the proof of theorem3.1.

Theorem 3.2. Let $f_\alpha(z)\in T_n^\alpha(a,c,\beta,\lambda,l)$ Then, for $\lambda\ge 0;\ l\ge 0;\ a\in\Re;\ c\in\Re-\{0,-1,-2,...\};$ $\alpha\in N;\ 0\le\beta<1;\ \mu\in\Re;\ n\in N_0$ *and* $z\in U$

$$\left|a_{\alpha+2}-\mu a_{\alpha+1}^2\right|\le\frac{1}{2\psi^2(a)_\alpha^2(a)_{\alpha+1}\gamma_1^2\gamma_2\phi_1^2\phi_2}\left|(a)_\alpha^2(c)_{\alpha+1}\gamma_1^2\phi_1^2\psi(\alpha p_1^2+p_2)-2\mu(a)_{\alpha+1}(c)_\alpha^2\gamma_2\phi_2 p_1^2\right|$$

where

$$\gamma_1=[1+\beta(1+\alpha)],\quad \gamma_2=[1+\beta(2+\alpha)],\quad \gamma_3=[1+\beta(3+\alpha)],\ \phi_1=\left(\frac{1+\alpha\lambda+l}{1+\lambda(\alpha-1)+l}\right)^n,$$

$$\phi_2=\left(\frac{1+\lambda(1+\alpha)+l}{1+\lambda(\alpha-1)+l}\right)^n,\quad \phi_3=\left(\frac{1+\lambda(2+\alpha)+l}{1+\lambda(\alpha-1)+l}\right)^n \quad and \quad \psi=\frac{1}{\alpha(1+\alpha\beta)}$$

Proof: The proof follows immediately from (18) and (19).

Next, we proceed to the Hankel determinant.

Theorem 3.3. Let $f_\alpha(z)\in T_n^\alpha(a,c,\beta,\lambda,l)$ Then, for $\lambda\geq 0;\ l\geq 0;\ a\in\Re;\ c\in\Re-\{0,-1,-2,\ldots\};$ $\alpha\in N; 0\leq\beta<1;\ n\in N_0\ and\ z\in U$

$$\left|a_{\alpha+1}a_{\alpha+3}-a_{\alpha+2}^2\right|\leq\frac{\alpha(c)_{\alpha+1}^2(1+\alpha\beta)}{(a)_{\alpha+1}^2\gamma_2^2\,\phi_2^2}$$

Where

$$\gamma_2=[1+\beta(2+\alpha)]\ and\ \phi_2=\left(\frac{1+\lambda(1+\alpha)+l}{1+\lambda(\alpha-1)+l}\right)^n.$$

Proof: As $f_\alpha(z)\in T_n^\alpha(a,c,\beta,\lambda,l)$ from theorem 3.1, using (18), (19) and (20) we have that,

$$a_{\alpha+1}a_{\alpha+3}-a_{\alpha+2}^2=\frac{1}{H(\alpha)}\left[(M-N)\alpha^2\,p_1^4+(3M-2N)\alpha\,p_1^2p_2+2M\,p_1p_3-Np_2^2\right]\quad(21)$$

where

$$M=2(c)_\alpha(c)_{\alpha+2}(a)_{\alpha+1}^2\,\gamma_2^2\,\phi_2^2,$$
$$N=3(a)_\alpha(a)_{\alpha+2}(c)_{\alpha+1}^2\,\gamma_1\,\gamma_3\,\phi_1\,\phi_3,$$
$$H(\alpha)=12(a)_\alpha(a)_{\alpha+2}(a)_{\alpha+1}^2\psi^2\,\gamma_1\,\gamma_2^2\,\gamma_3\,\phi_1\,\phi_2^2\,\phi_3$$

and $\gamma_1,\gamma_2,\gamma_3,\phi_1,\phi_2,\phi_3$ are as earlier defined.

Now, using lemma 2.1 and lemma 2.2, we obtain

$$\left|a_{\alpha+1}a_{\alpha+3}-a_{\alpha+2}^2\right|\leq\frac{1}{4H(\alpha)}\left|\begin{array}{l}4\alpha^2(M-N)p_1^4+2\alpha(3M-2N)\left(p_1^4+p_1^2(4-p_1^2)x\right)\\+2M\left(p_1^4+2p_1^2(4-p_1^2)x-p_1^2(4-p_1^2)x^2+2p_1(4-p_1^2)(1-|x|^2)z\right)\\-N\left(p_1^2+(4-p_1^2)x\right)^2\end{array}\right|.$$

Assuming that $p_1=p\ and\ p\in[0,2],$ using triangular inequality and $|z|\leq 1$, we obtain

$$\left|a_{\alpha+1}a_{\alpha+3}-a_{\alpha+2}^{2}\right|\le\frac{1}{4H(\alpha)}\left\{\begin{array}{l}\left[4\alpha^{2}(M-N)+2\alpha(3M-2N)+2M-N\right]p^{4}\\+\left[2\alpha(3M-2N)+4M-2N\right]p^{2}(4-p^{2})\delta-\left[2Mp^{2}+N(4-p^{2})\right](4-p^{2})\delta^{2}\\4M\ p\ (4-p^{2})-4M\ p\ (4-p^{2})\delta^{2}\end{array}\right\}$$

$$=\frac{1}{4H(\alpha)}F(\delta) \tag{22}$$

where, $\delta=|x|\le 1$ and

$$F(\delta)=\left[4\alpha^{2}(M-N)+2\alpha(3M-2N)+2M-N\right]p^{4}+\left[2\alpha(3M-2N)+4M-2N\right]p^{2}(4-p^{2})\delta$$
$$-\left[2Mp^{2}+N(4-p^{2})\right](4-p^{2})\delta^{2}+4M\ p\ (4-p^{2})-4M\ p\ (4-p^{2})\delta^{2}$$

is an increasing function.

Therefore,

$$Max F(\delta)=F(1).$$

Consequently,

$$\left|a_{2}(\alpha)a_{4}(\alpha)-a_{3}^{2}(\alpha)\right|\le\frac{1}{4H(\alpha)}G(p) \tag{23}$$

where

$$G(p)=F(1).$$

So

$$G(p)=2\left[2\alpha^{2}(M-N)-2M+N\right]p^{4}+8\left[\alpha(3M-2N)+3M-2N\right]p^{2}+16N,$$

$$G'(p)=8\left[2\alpha^{2}(M-N)-2M+N\right]p^{3}+16\left[\alpha(3M-2N)+3M-2N\right]p$$

and

$$G''(p)=24\left[2\alpha^{2}(M-N)-2M+N\right]p^{2}+16\left[\alpha(3M-2N)+3M-2N\right].$$

Also,

$$G'(p)=0 \quad implies \quad that \quad 2p[2L(\alpha)p^{2}-M(\alpha)]=0.$$

Obviously, *G(p)* attains its maximum value at *p=0*. Thus,

$$Max.\,G(p)=G(0).$$

Hence, from (23), we obtain our desired result. Finally, with various choices of parameters involved, several results (known and new) are obtained. Few of them are given below.

Suppose that $\alpha = 1$ in theorem 3.3, then we obtain the following corollary.

Corollary 3.4. Let $f_\alpha \in T_n^1(a, c, \beta, \lambda, l)$. Then, for $\lambda \geq 0; l \geq 0;\ a \in \Re;\ c \in \Re - \{0, -1, -2, ...\}$;

$\alpha \in N;\ 0 \leq \beta < 1;\ n \in N_0;\ and\ \ z \in U$

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{(c)_2^2 (1+\beta)}{(a)_2^2 (1+2\beta)^2 \left(\dfrac{1+2\lambda+l}{1+l}\right)^{2n}}.$$

Also, if $\alpha = 1$ and $n = 0$ in theorem 3.3, then we obtain the following corollary.

Corollary 3.5. Let $f_\alpha \in T_0^1(a, c, \beta)$. Then, for $\lambda \geq 0; l \geq 0; \alpha \in N;\ a \in \Re;\ c \in \Re - \{0, -1, -2, ...\}$; $0 \leq \beta < 1;\ n \in N_0;\ and\ \ z \in U$

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{(c)_2^2 (1+\beta)}{(a)_2^2 (1+2\beta)^2}.$$

Finally, suppose that $\alpha = 1$, $n = \beta = 0$ and $a = c$ in theorem 3.3, then we obtain the following corollary.

Corollary 3.6. Let $f_\alpha \in T_0^1(a, a, 0)$. Then, for $\lambda \geq 0; l \geq 0; \alpha \in N;\ a \in \Re;\ c \in \Re - \{0, -1, -2, ...\}$; $0 \leq \beta < 1;\ n \in N_0;\ and\ \ z \in U$

$$\left|a_2 a_4 - a_3^2\right| \leq 1.$$

Incidentally, the result in corollary 3.6 coincides with that of Janteng [18], Theorem 3.1.

Competing Interests

Authors have declared that no competing interests exist.

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