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On a New Class of Weakly Berwald Spaces with (α, β)-metric

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Authors' contributions

This work was carried out in collaboration between both authors. Author GS presuming the concept and helped out determining the metrics to be worked upon. Author DC worked out the approach through calculation and managed the analyses of the study with literature searches. Both authors read and approved the final manuscript.

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Abstract

We have two concepts of Douglas spaces and Landsberg spaces as generalizations of Berwald spaces. S. Bacso [1] gave the definition of a weakly-Berwald space as another generalization of Berwald spaces. In 1972, M. Matsumoto has introduced the concept of (α, β)-metric, which is a Finsler meric, contstructed from a Riemannian metric and a differential 1-form. In this paper, we study an important class of (α, β)-metrics in the form $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$, known as second approximate Matsumoto metric on an n-dimensional manifold and get the conditions for such metrics to be weakly-Berwald metrics, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form. A Finsler space with an (α, β)-metric is a weakly-Berwald space, if and only if $B^m_m = \partial B^m / \partial y^m$ is a 1-form. We show that it becomes a weakly Berwald space under some geometric and algebraic conditions.

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1 Introduction

Let M^n be an n-dimensional differential manifold and let $F^n = (M^n, L)$ be an n-dimensional Finsler space where L is a fundamental function. Let $g_{ij} = \dot{\partial}_i\dot{\partial}_j L^2/2$ be the fundamental tensor, where the symbol $\dot{\partial}_i$ means $\frac{\partial}{\partial y^i}$ and we define G_i as

$$G_i = \{y^r(\partial_r\dot{\partial}_i L^2) - \partial_i L^2\}/4,$$

and $G^i = g^{ij}G_j$ where the symbol ∂_i means $\frac{\partial}{\partial x^i}$ and (g^{ij}) is the inverse matrix of (g_{ij}). The coefficients (G^i_{jk}, G^i_j) of the Berwald connection $B\Gamma$ are defined as $G^i_j = \dot{\partial}_j G^i$ and $G^i_{jk} = \dot{\partial}_k G^i_j$.

A Berwald space is a Finsler space which satisfies the condition $G^h_{ijk} = 0$, that is to say, whose coefficients G^h_{ij} of the Berwald connection are functions of the position (x^i) alone. Therefore the equations $y_r G^r_{ijk} = 0$ hold, so $2G^i = G^i_{rs}y^r y^s$ are homogeneous polynomials in (y^i) of degree two, so $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three. Then we can consider the notions of Landsberg spaces and Douglas spaces as two generalizations of Berwald spaces. The notion of weakly-Berwald spaces is the third generalization of Berwald spaces. Thus if a Finsler space satisfies the condition $G_{ij} = 0$, we call it a weakly-Berwald space.

The concept of an (α,β)-metric on a Finsler space $F^n = (M^n, L)$ was introduced by M. Matsumoto in [2] and has been studied by many authors [3], [4], [5], [6]. Some important examples of (α,β)-metric are Randers metric $(L = \alpha + \beta)$, Kropina metric $\left(L = \dfrac{\alpha^2}{\beta}\right)$, generalized Kropina metric $\left(L = \dfrac{\alpha^{m+1}}{\beta^m}\right)$, $(m \neq 0, -1)$, and Matsumoto metric $\left(L = \frac{\alpha^2}{\alpha-\beta}\right)$. The study of Finsler spaces with these metrics has greatly contributed to the growth of Finsler geometry and its applications to the theory of Relativity and allied areas.

Definition 1.1. A Finsler metric $L(x,y)$ is called an (α,β)-metric if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n.

S. Bacso [1] introduced the notion of weakly-Berwald space as another generalization of Berwald spaces. IL-Yong Lee and Myung-Han Lee [7] have studied weakly Berwald spaces with special (α,β)-metrics. In this paper we extend the study on weakly Berwald spaces with second approximate Matsumoto metric.

Let $F^n = (M^n, L)$ be a Finsler space of dimension n, and the domain of the fundamental metric function $L(x,y)$ is the set of $TM\setminus(0)$ of the non- zero tangent vectors. We assume that L is positive and the fundamental metric tensor $g_{ij}(x,y) = \dfrac{1}{2}L^2_{(i)(j)}$ (where $(i) := \dfrac{\partial}{\partial y^i}$) is not necessarily positive definite.

The equation of the (canonically parametrized) geodesics of F^n is given by

$$\frac{d^2x^i}{dt^2} + 2G^i(x,y) = 0,\ \left(\frac{dx^i}{dt} = y^i\right),$$

where

$$G^i = \frac{1}{2} g^{ir} \left(y^s \frac{\partial L_r^2}{\partial x^s} - \frac{\partial L^2}{\partial x^r} \right).$$

The Berwald connection of the space is defined by its connection coefficients $G^i_{jk}(x, y)$ which can be computed from G^i according to the following formula:

$$G^i_j(x, y) = G^i_{(j)};\ G^i_{jk}(x, y) = G^i_{j(k)}.$$

Definition 1.2. A Finsler space is called Berwald space, if G^i_{jk} are functions of position alone, i. e., Berwald connection $B\Gamma$ is linear.

A Finsler space is called a weakly Berwald space if the (hv)-Ricci curvature tensor $G_{jk} = 0$. The spray function G^i of a Finsler space with an (α, β)-metric are given by $2G^i = \gamma^i_{00} + 2B^i$, where γ^i_{jk} stands for the Christoffel symbols in the associated Riemannian space (M^n, α).Then we have $G^i_j = \gamma^i_{0j} + B^i_j$ and $G^i_{jk} = \gamma^i_{jk} + B^i_{jk}$, where $\dot{\partial}_j B^i = B^i_j$ and $\dot{\partial}_k B^i_j = B^i_{jk}$. Thus a Finsler space with an (α, β)-metric is a weakly-Berwald space, if and only if $B^m_m = \partial B^m / \partial y^m$ is a one-form.

Recently, S. Bacso and B. Szilagyi [1] gave an example for the weakly Berwald Finsler space, and a sufficient condition for the existence of a weakly Berwald Finsler space of Kropina type was also determined. Recently, S. Bacso and R. Yoshikawa [8] investigated the conditions for Randers and Kropina spaces to be weakly-Berwald spaces. R. Yoshikawa and K. Okubo [9] obtained the conditions for generalized Kropina spaces and Matsumoto spaces to be weakly-Berwald spaces and Berwald spaces. In [10], it has been shown under which condition a Finsler space with first approximate Matsumoto metric becomes a weakly Berwald space. In this paper, first we discuss the conditions for the Finsler space F^n with an (α, β)-metric to be a weakly-Berwald space and then we find the conditions for Finsler space with second approximate Matsumoto metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ to be weakly-Berwald space.

2 Weakly-Berwald Space with Respect to (α, β)-metric

In the present section, we deal with the condition that a Finsler space with an (α, β)-metric be a weakly-Berwald space.

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with (α, β)-metric $L(\alpha, \beta)$, where $\alpha = \sqrt{(a_{ij}(x) y^i y^j)}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form. In this paper, the symbol $(;)$ stands for h-covariant derivation with respect to the Riemannian connection in the associated Riemannian space (M^n, α) and γ^i_{jk} stands for the Christoffel symbols in the space (M^n, α).
We use the following notations [8]

$$(a)\ b^i = a^{ir} b_r,\ b^2 = a^{rs} b_r b_s,$$
$$(b)\ 2r_{ij} = b_{i;j} + b_{j;i},\ 2s_{ij} = b_{i;j} - b_{j;i},$$
$$(c)\ r^i_j = a^{ir} r_{rj},\ s^i_j = a^{ir} s_{rj},\ r_i = b_r r^r_i,\ s_i = b_r s^r_i.$$

We now consider the function $G^i(x, y)$ of F^n with an (α, β)-metric. According to [11], they are written in the form

$$2G^m = \gamma^m_{00} + 2B^m,$$
$$B^m = \frac{E^*}{\alpha} y^m + \frac{\alpha L_\beta}{L_\alpha} s^m_0 - \frac{\alpha L_{\alpha\alpha}}{L\alpha} C^* \left(\frac{1}{\alpha} y^m - \frac{\alpha}{\beta} b^m \right), \tag{2.1}$$

where we have put

$$E^* = \left(\frac{\beta L_\beta}{L} \right) C^*, C^* = \frac{\alpha\beta(r_{00} L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}, \gamma^2 = b^2\alpha^2 - \beta^2, \tag{2.2}$$

and

$$L_\alpha = \frac{\partial L}{\partial \alpha},\ L_\beta = \frac{\partial L}{\partial \beta},\ L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2},\ L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta},\ L_{\alpha\alpha\alpha} = \frac{\partial^3 L}{\partial \alpha^3}. \tag{2.3}$$

Since $\gamma^i_{00} = \gamma^i_{jk}(x) y^j y^k$ are homogeneous polynomials in (y^i) of degree two, it is obvious that a Finsler space with an (α, β)-metric is a Berwald space, if and only if B^m are homogeneous polynomials in (y^i) of degree two and Berwald connection $B\Gamma$ is linear.

Differentiating (2.1) with respect to y^n and contracting m and n in the obtained equation, we have

$$\begin{aligned} B^m_m = & \left\{ \dot{\partial}_m \left(\frac{\beta L_\beta}{\alpha L} \right) y^m + \frac{n \beta L_\beta}{\alpha L} - \dot{\partial}_m \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha} \right) \left(\frac{\beta y^m - \alpha^2 b^m}{\alpha \beta} \right) \right\} C^* \\ & - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left\{ \dot{\partial}_m \left(\frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \delta^m_m - \dot{\partial}_m \left(\frac{\alpha}{\beta} \right) b^m \right\} C^* + \dot{\partial}_m \left(\frac{\alpha L_\beta}{L_\alpha} \right) s^m_0 \\ & + \left(\frac{\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}}{\alpha L L_\alpha} \right) \left(\dot{\partial}_m C^* \right) y^m + \left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} \right) \left(\dot{\partial}_m C^* \right) b^m. \end{aligned} \tag{2.4}$$

Since $L = L(\alpha, \beta)$ is a positively homogeneous function of α and β of degree one, we have

$$L_\alpha \alpha + L_\beta \beta = L,\ L_{\alpha\alpha} \alpha + L_{\alpha\beta} \beta = 0,$$

$$L_{\beta\alpha} \alpha + L_{\beta\beta} \beta = 0,\ L_{\alpha\alpha\alpha} \alpha + L_{\alpha\alpha\beta} \beta = -L_{\alpha\alpha}.$$

Using the above relations and the homogeneity of (y^i), we obtain the following equations

$$\dot{\partial}_m \left(\frac{\beta L_\beta}{\alpha L} \right) y^m = -\frac{\beta L_\beta}{\alpha L}, \tag{2.5}$$

$$\dot{\partial}_m \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha} \right) \left(\frac{\beta y^m - \alpha^2 b^m}{\alpha \beta} \right) = \frac{\gamma^2}{(\beta L_\alpha)^2} \left\{ L_\alpha L_{\alpha\alpha} + \alpha L_\alpha L_{\alpha\alpha\alpha} - \alpha (L_{\alpha\alpha})^2 \right\}, \tag{2.6}$$

$$\left\{ \dot{\partial}_m \left(\frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \delta^m_m - \dot{\partial}_m \left(\frac{\alpha}{\beta} \right) b^m \right\} = \frac{1}{\alpha \beta^2} \left(\gamma^2 + (n-1) \beta^2 \right), \tag{2.7}$$

$$\left(\dot{\partial}_m C^* \right) y^m = 2C^*, \tag{2.8}$$

$$\begin{aligned} \left(\dot{\partial}_m C^* \right) b^m = \frac{1}{2\alpha\beta\Omega^2} \Bigg(& \Omega \Big(\beta(\gamma^2 + 2\beta^2) W + 2\alpha^2 \beta^2 L_\alpha r_0 - \\ & \alpha \beta \gamma^2 L_{\alpha\alpha} r_{00} - 2\alpha (\beta^3 L_\beta + \alpha^2 \gamma^2 L_{\alpha\alpha}) s_0 \Big) - \\ & \alpha^2 \beta W \left(2b^2 \beta^2 L_\alpha - \gamma^4 L_{\alpha\alpha\alpha} - b^2 \alpha \gamma^2 L_{\alpha\alpha} \right) \Bigg), \end{aligned} \tag{2.9}$$

$$\dot{\partial}_m \left(\frac{\alpha L_\beta}{L_\alpha} \right) s^m_0 = \frac{\alpha^2 L L_{\alpha\alpha} s_0}{(\beta L_\alpha)^2}, \tag{2.10}$$

where

$$W = (r_{00} L_\alpha - 2\alpha s_0 L_\beta), \Omega = (\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha}),\ (\Omega \neq 0) \tag{2.11}$$

$$Y_i = a_{ir} y^r,\ s_{00} = 0,\ b^r s_r = 0,\ a^{ij} s_{ij} = 0.$$

Substituting (2.2), (2.3), (2.5), (2.6), (2.7), (2.8), (2.9) and (2.10) into (2.4), we get

$$B^m_m = \frac{1}{2\alpha L (\beta L_\alpha)^2 \Omega^2} \{ 2\Omega^2 A C^* + 2\alpha L \Omega^2 B s_0 + \alpha^2 L_\alpha L_{\alpha\alpha} (C r_{00} + D s_0 + E r_0) \}, \tag{2.12}$$

where

$$\begin{aligned}
A &= (n+1)\beta^2 L_\alpha(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}) + \alpha\gamma^2 L(\alpha(L_{\alpha\alpha})^2 \\
&\quad - 2L_\alpha L_{\alpha\alpha} - \alpha L_\alpha L_{\alpha\alpha\alpha}), \\
B &= \alpha^2 L L_{\alpha\alpha}, \\
C &= \beta\gamma^2\left(-\beta^2(L_\alpha)^2 + 2b^2\alpha^3 L_\alpha L_{\alpha\alpha} - \alpha^2\gamma^2(L_{\alpha\alpha})^2 + \alpha^2\gamma^2 L_\alpha L_{\alpha\alpha\alpha}\right), \\
D &= 2\alpha(\beta^3(\gamma^2-\beta^2)L_\alpha L_\beta - \alpha^2\beta^2\gamma^2 L_\alpha L_{\alpha\alpha} \\
&\quad - 2\alpha\beta\gamma^2(\gamma^2+2\beta^2)L_\beta L_{\alpha\alpha} - \alpha^3\gamma^4(L_{\alpha\alpha})^2 - \alpha^2\beta\gamma^4 L_\beta L_{\alpha\alpha\alpha}), \\
E &= 2\alpha^2\beta^2 L_\alpha \Omega. \qquad (2.13)
\end{aligned}$$

Summarizing up the above, we have the following

Theorem 2.1. The necessary and sufficient condition for a Finsler space F^n with an (α,β)-metric to be a weakly-Berwald space is that $G^m_m = \gamma^m_{0m} + B^m_m$ and B^m_m is a homogeneous polynomial in (y^m) of degree one, where B^m_m is given by (2.12) and (2.13), provided that $\Omega \neq 0$.

Lemma 2.2. If $\alpha^2 \equiv 0(mod, \beta)$, that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension n is equal to 2 and b^2 vanishes. In this case we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.

3 Finsler Space with Second Approximate Matsumoto Metric

In the present section, we consider the condition that the Finsler space with second approximate Matsumoto metric be a weakly-Berwlad space.

Let us consider a Finsler space $F^n = (M^n, L)$ with second approximate Matsumoto metric

$$L(\alpha,\beta) = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}. \qquad (3.1)$$

We now find the conditions for F^n with (3.1) to be a weakly-Berwald space. For F^n with (3.1), we have

$$\begin{aligned}
L_\alpha &= \frac{\alpha^3 - \alpha\beta^2 - 2\beta^3}{\alpha^3},\ L_\beta = \frac{\alpha^2 + 2\alpha\beta + 3\beta^2}{\alpha^2}, \\
L_{\alpha\alpha} &= \frac{2\beta^2(\alpha+3\beta)}{\alpha^4},\ L_{\alpha\alpha\alpha} = -\frac{6\beta^2(\alpha+4\beta)}{\alpha^5}. \qquad (3.2)
\end{aligned}$$

Substituting (3.2) into (2.1), we get

$$\begin{aligned}
B^m &= \frac{r_{00}(\alpha^3 - \alpha\beta^2 - 2\beta^3) - 2s_0\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)}{(\alpha^3 - \alpha\beta^2 - 2\beta^3) + 2(b^2\alpha^2 - \beta^2)(\alpha+3\beta)} \\
&\left(\left(\frac{(\alpha^2 + 2\alpha\beta + 3\beta^2)}{2(\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3)} - \frac{\beta(\alpha+3\beta)}{(\alpha^3 - \alpha\beta^2 - 2\beta^3)}\right)y^m \right. \qquad (3.3) \\
&\left. - \frac{\alpha^2(\alpha+3\beta)}{(\alpha^3 - \alpha\beta^2 - 2\beta^3)}b^m\right) + \frac{\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)}{(\alpha^3 - \alpha\beta^2 - 2\beta^3)}s^m_0.
\end{aligned}$$

Again, substituting (3.2) into (2.2), (2.11), and (2.13) in respective quantities, we obtain

$$
\begin{aligned}
A &= (n+1)\frac{\beta^3}{\alpha^8}(\alpha^3-\alpha\beta^2-2\beta^3)(\alpha^5-6\alpha^3\beta^2-12\alpha^2\beta^3-15\alpha\beta^4-12\beta^5)\\
&\quad +2\frac{\beta^2}{\alpha^8}(b^2\alpha^2-\beta^2)(\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3)(\alpha^4+6\alpha^3\beta+\alpha^2\beta^2+4\alpha\beta^3+6\beta^4),\\
B &= 2\frac{\beta^2}{\alpha^4}(\alpha+3\beta)(\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3),\\
C &= \frac{\beta^3}{\alpha^6}(b^2\alpha^2-\beta^2)\Big((4b^2-1)\alpha^6-12b^2\alpha^5\beta-2(4-b^2)\alpha^4\beta^2\\
&\quad -4(2b^2+5)\alpha^3\beta^3+3(36b^2-1)\alpha^2\beta^4+16\alpha\beta^5-16\beta^6\Big),\\
D &= 2\frac{\beta^3}{\alpha^4}\Big(b^2(1+2b^2)\alpha^7+12b^4\alpha^6\beta+2(3b^4-8b^2-1)\alpha^5\beta^2\\
&\quad -2(33b^2+1)\alpha^4\beta^3+(36-41b^2)\alpha^3\beta^4\\
&\quad +2(29-33b^2)\alpha^2\beta^5+82\alpha\beta^6+72\beta^7\Big),\\
E &= 2\frac{\beta^4}{\alpha^4}\Big((1+2b^2)\alpha^6+6b^2\alpha^5\beta-(2b^2+4)\alpha^4\beta^2-10(b^2+1)\alpha^3\beta^3\\
&\quad -3(4b^2-1)\alpha^2\beta^4+14\alpha\beta^5+16\beta^6\Big),\\
W &= \frac{r_{00}(\alpha^3-\alpha\beta^2-2\beta^3)-2\alpha^2 s_0(\alpha^2+2\alpha\beta+3\beta^2)}{\alpha^3},\\
\Omega &= \frac{\beta^2}{\alpha^3}\Big(\alpha^3(1+2b^2)+2b^2\alpha^2\beta-3\alpha\beta^2-8\beta^3\Big),\\
C^* &= \frac{\alpha}{2\beta}\left(\frac{r_{00}(\alpha^3-\alpha\beta^2-2\beta^3)-2s_0\alpha^2(\alpha^2+2\alpha\beta+3\beta^2)}{(\alpha^3-\alpha\beta^2-2\beta^3)+2(b^2\alpha^2-\beta^2)(\alpha+3\beta)}\right),\\
E^* &= \frac{\alpha(\alpha^2+2\alpha\beta+3\beta^2)}{2(\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3)}\\
&\quad \left(\frac{r_{00}(\alpha^3-\alpha\beta^2-2\beta^3)-2s_0\alpha^2(\alpha^2+2\alpha\beta+3\beta^2)}{(\alpha^3-\alpha\beta^2-2\beta^3)+2(b^2\alpha^2-\beta^2)(\alpha+3\beta)}\right),
\end{aligned}
\tag{3.4}
$$

Substituting (3.4) into (2.12), we get

$$
\begin{aligned}
&2B^m_m\Big((1+2b^2)^2\alpha^{15}\beta+a_1\alpha^{14}\beta^2+a_2\alpha^{13}\beta^3-a_3\alpha^{12}\beta^4-2a_4\alpha^{11}\beta^5\\
&+2a_5\alpha^{10}\beta^6+2a_6\alpha^9\beta^7+2a_7\alpha^8\beta^8+a_8\alpha^7\beta^9+a_9\alpha^6\beta^{10}+a_{10}\alpha^5\beta^{11}\\
&+a_{11}\alpha^4\beta^{12}+40(19-26b^2)\alpha^3\beta^{13}+12(83-32b^2)\alpha^2\beta^{14}+704\alpha\beta^{15}+256\beta^{16}\Big)\\
&-r_{00}\Big(12b^4\alpha^{15}+a_{12}\alpha^{14}\beta+2b^2a_{13}\alpha^{13}\beta^2+a_{14}\alpha^{12}\beta^3+2a_{15}\alpha^{11}\beta^4+2a_{16}\alpha^{10}\beta^5\\
&+2a_{17}\alpha^9\beta^6+2a_{18}\alpha^8\beta^7+4a_{19}\alpha^7\beta^8+a_{20}\alpha^6\beta^9+2a_{21}\alpha^5\beta^{10}+a_{22}\alpha^4\beta^{11}\\
&+2a_{23}\alpha^3\beta^{12}+2a_{24}\alpha^2\beta^{13}-160\alpha\beta^{14}-192\beta^{15}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- 2s_0\Big(a_{25}\alpha^{15}\beta + 2a_{26}\alpha^{14}\beta^2 + a_{27}\alpha^{13}\beta^3 + 2a_{28}\alpha^{12}\beta^4 + 2a_{29}\alpha^{11}\beta^5\\
&+ 2a_{30}\alpha^{10}\beta^6 + 2a_{31}\alpha^9\beta^7 + 6a_{32}\alpha^8\beta^8 + a_{33}\alpha^7\beta^9 + 2a_{34}\alpha^6\beta^{10} + a_{35}\alpha^5\beta^{11}\\
&+ 6a_{36}\alpha^4\beta^{12} - 1384\alpha^3\beta^{13} - 864\alpha^2\beta^{14}\Big)\\
&- 2r_0\Big(2(1+2b^2)\alpha^{15}\beta + 4(7b^2+2)\alpha^{14}\beta^2 + 2(28b^2-1)\alpha^{13}\beta^3\\
&- 8(b^2+7)\alpha^{12}\beta^4 - 4(44b^2+29)\alpha^{11}\beta^5 - 12(32b^2-3)\alpha^{10}\beta^6\\
&+ 4(-46b^2+63)\alpha^9\beta^8 + 4(50b^2+87)\alpha^8\beta^8 + 2(334b^2-17)\alpha^7\beta^9\\
&+ 4(133b^2-30)\alpha^6\beta^{10} + 6(64b^2-95)\alpha^5\beta^{11} + 8(18b^2-91)\alpha^4\beta^{12}\\
&- 640\alpha^3\beta^{13} - 192\alpha^2\beta^{14}\Big) = 0, \qquad (3.5)
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= (28b^4+16b^2+1),\ a_2 = (56b^4-4b^2-7),\\
a_3 &= (8b^4+112b^2+27),\ a_4 = (88b^4+116b^2+3),\\
a_5 &= (51-16b^2-152b^4),\ a_6 = (109+252b^2-92b^4),\\
a_7 &= (33+464b^2+60b^4),\ a_8 = (428b^4+628b^2-355),\\
a_9 &= (532b^4-240b^2-699),\ a_{10} = (384b^4-1140b^2-515),\\
a_{11} &= (144b^4-1156b^2+105),\ a_{12} = (48b^4+8b^2+2b^2n+n+1),\\
a_{13} &= (22b^2-10+3n),\ a_{14} = (8b^4-120b^2-16b^2n-11n-17),\\
a_{15} &= (78b^4-26b^2-40b^2n-8n-1),\ a_{16} = (412b^4-50b^2n+11n+79),\\
a_{17} &= (294b^4-17b^2+34b^2n+44n+168),\\
a_{18} &= (-324b^4-250b^2+196b^2n+41n+95),\\
a_{19} &= (-562b^4+87b^2+138b^2n-18n-91),\\
a_{20} &= (1952b^4+1356b^2+310b^2n-247n-835),\\
a_{21} &= (-1320b^4+164b62-237b62n-168n-468),\\
a_{22} &= (-1440b^4+2216b^2-840b^2n+201n-557),\\
a_{23} &= (1064b^2-372b^2n+156n-33),\\
a_{24} &= (672b^2-144b^2n+36n-16),\\
a_{25} &= (-4b^2-2b^2n-n+1),\ a_{26} = (-8b^2-5nb^2-n+1),\\
a_{27} &= (-24b^4-96b^2-4b^2n+7n+11),\\
a_{28} &= (-12b^4-180b^2+40b^2n+17n+11),\\
a_{29} &= (-555b^4-280b^2+142b^2n+23n+63),\\
a_{30} &= (587b^4b^4-387b^2+222b^2n-18n+298),\\
a_{31} &= (-334b^4-854b^2+108b^2n-105n+467),\\
a_{32} &= (-12b^4-79b^2-88b^2n-29n+193),
\end{aligned}
$$

$$a_{33} = (72b^{-}564b^2 - 1386b^2n - 117n + 13),$$
$$a_{34} = (-214b^2 - 849b^2n + 147n - 1903),$$
$$a_{35} = (-312b^2 - 1188b^2n + 603n + 631), a_{36} = (-72b^2n + 87n - 304).$$

Now we assume that F^n is a weakly Berwald space, then B^m_m is hp(1). Since α is irrational in (y^i), the equation (3.5) is divided into two equations as follows,

$$\beta^2 B^m_m F_1 + \beta r_{00} G_1 + \alpha^2\beta^2 s_0 H_1 + \alpha^2\beta^2 r_0 I_1 = 0, \tag{3.6}$$

$$\alpha\beta B^m_m F_2 + \alpha r_{00} G_2 + \alpha^3\beta s_0 H_2 + \alpha^3\beta r_0 I_2 = 0, \tag{3.7}$$

where

$$F_1 = 2a_1\alpha^{14} - 2a_3\alpha^{12}\beta^2 + 4a_5\alpha^{10}\beta^4 + 4a_7\alpha^8\beta^6 + 2a_9\alpha^6\beta^8$$
$$+ 2a_{11}\alpha^4\beta^{10} + 24(83 - 32b^2)\alpha^2\beta^{12} + 512\beta^{14},$$
$$F_2 = 2a_1\alpha^{14} + 2a_2\alpha^{12}\beta^2 - 4a_4\alpha^{10}\beta^4 + 4a_6\alpha^8\beta^6 + 2a_8\alpha^6\beta^8$$
$$+ 2a_{10}\alpha^4\beta^{10} + 80(19 - 26b^2)\alpha^2\beta^{12} + 1408b^{14},$$
$$G_1 = -(a_{12}\alpha\alpha^{14} + a_{14}\alpha^{12}\beta^2 + 2a_{16}\alpha^{10}\beta^4 + 2a_{18}\alpha^8\beta^6 + a_{20}\alpha^6\beta^8 + a_{22}\alpha^4\beta^{10}$$
$$+ 2a_{24}\alpha^2\beta^{12} - 192\beta^{14}),$$
$$G_2 = -2(6b^4\alpha^{14} + b^2a_{13}\alpha^{12}\beta^2 + a_{15}\alpha^{10}\beta^4 + a_{17}\alpha^8\beta^6 + 2a_{19}\alpha^6\beta^8 + a_{21}\alpha^4\beta^{10}$$
$$+ a_{23}\alpha^2\beta^{12} - 80\beta^{14}),$$
$$H_1 = -4(a_{26}\alpha^{12} + a_{28}\alpha^{10}\beta^2 + a_{30}\alpha^8\beta^4 + 3a_{32}\alpha^6\beta^6 + a_{34}\alpha^4\beta^8 + 3a_{36}\alpha^2\beta^{10}$$
$$- 432\beta^{12}),$$
$$H_2 = -2(a_{25}\alpha^{12} + a_{27}\alpha^{10}\beta^2 + 2a_{29}\alpha^8\beta^4 + 2a_{31}\alpha^6\beta^6 + a_{33}\alpha^4\beta^8 + a_{35}\alpha^2\beta^{10}$$
$$- 1384\beta^{12}),$$
$$I_1 = -8\{(7b^2 + 2)\alpha^{12} - 2(b^2 + 7)\alpha^{10}\beta^2 - 3(32b^2 - 3)\alpha^8\beta^4$$
$$+ (50b^2 + 87)\alpha^6\beta^6 + (133b^2 - 30)\alpha^4\beta^8 + 2(18b^2 - 91)\alpha^2\beta^{10} - 48\beta^{12}\},$$
$$I_2 = -4\{(1 + 2b^2)\alpha^{12} + (28b^2 - 1)\alpha^{10}\beta^2 - 2(44b^2 + 29)\alpha^8\beta^4$$
$$+ 2(-46b^2 + 63)\alpha^6\beta^6 + (334b^2 - 17)\alpha^4\beta^8$$
$$+ 3(64b^2 - 95)\alpha^2\beta^{10} - 320\beta^{12}\}.$$

Eliminating B^m_m from these equations, we obtain

$$Rr_{00} + \alpha^2\beta S_0 + \alpha^2\beta T r_0 = 0, \tag{3.8}$$

where

$$R = \beta^2 F_2 G_1 - F_1 G_2,\ S = F_2 H_1 - F_1 H_2,\ T = I_1 F_2 - I_2 F_1.$$

From (3.8),we get

$$(R/\alpha^2\beta) r_{00} + S s_0 + T r_0 = 0. \tag{3.9}$$

Since only the term $\epsilon_1\alpha^{26}$ of Ss_0 in (3.9) does not contain β, we must have $hp(26)V_{26}$, such that

$$\alpha^{26} s_0 = \beta V_{26}, \tag{3.10}$$

where

$$\epsilon_1 = -4a_1(a_{26} - a_{25}).$$

First consider that $\alpha^2 \neq 0(mod\beta)$ and $b^2 \neq 0$. (3.10) shows the existence of a function $k(x)$ satisfying $V_{26} = k\alpha^{26}$, and hence $s_0 = k\beta$, (3.9) reduces to

$$(R/\alpha^2\beta)r_{00} + Sk\beta + Tr_0 = 0,$$

which implies that

$$Rr_{00} + Sk\alpha^2\beta^2 + \alpha^2\beta Tr_0 = 0.$$

Only the term $-2a_1\alpha^{28}(a_{12} - 12b^4)r_{00}$ of the above does not contain β. Thus there exist $hp(29)U_{29}$ satisfying $-2a_1\alpha^{28}(a_{12} - 12b^4)r_{00} = \beta U_{29}$. It is a contradiction, which implies $k = 0$. Hence we obtain $s_0 = 0; s_j = 0$. Therefore, (3.8) becomes

$$Rr_{00} + \alpha^2\beta Tr_0 = 0. \tag{3.11}$$

Only the term $188416\beta^{28}r_{00}$ of (34) seemingly does not contain α^2, and hence we must have $hp(28)V_{28}$ such that $\beta^{28}r_{00} = \alpha^2 V_{28}$. From $\alpha^2 \neq 0(mod\beta)$ there exist a function $f(x)$ such that

$$r_{00} = \alpha^2 f(x);\ r_{ij} = a_{ij}f(x). \tag{3.12}$$

Transvecting above by $b^i y^j$, we have

$$r_0 = \beta f(x);\ r_j = b_j f(x). \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11), we have

$$f(x)(R + \beta^2 T) = 0. \tag{3.14}$$

Assume that $f(x) \neq 0$, from (3.14) we get

$$R + \beta^2 T = 0,$$

The term $-2a_1\alpha^{28}(a_{12}-12b^4)$ of the above does not contain β. Thus there exist $hp(27)V_{27}$ satisfying $-2a_1\alpha^{28}(a_{12} - 12b^4) = \beta V_{27}$, where V_{27} is $hp(27)$ this implies $V_{27} = 0$, provided that $b^2 \neq 0$. Hence $f(x) = 0$ must hold and we obtain

$$r_{00} = 0;\ r_{ij} = 0\ and\ r_0 = 0;\ r_j = 0.$$

Conversely, substituting $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$ into (3.5), we have $B^m_m = 0$, that is, the Finsler space with (3.1) is a weakly-Berwald space.

On the other hand, if we suppose that the Finsler space with (3.1) is a Berwald space, then we have $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$, because the space is weakly Berwald space from the above discussion. Hence $s_{ij} = 0$ hold good.

Now consider $\alpha^2 \equiv 0(mod\beta)$, Lemma (2.2) shows that $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, $\delta = d_i(x)y_i$. From these conditions (3.8) is rewritten in the form

$$R^{'}r_{00} + \beta\delta S^{'}s_0 = 0, \tag{3.15}$$

where

$$
\begin{aligned}
R^{'} &= a_1a_{12}\beta^{14}\delta^{14} + (a_1a_{14} + a_2a_{12})\beta^{15}\delta^{13} + (2a_1a_{16} + a_2a_{14} - 2a_4a_{12} \\
&- 2a_1a_{15})\beta^{16}\delta^{12} + 2(a_1a_{18} + a_2a_{16} - a_4a_{14} + a_{12}a_6 - a_1a_{17} + a_3a_{15})\beta^{17}\delta^{11} \\
&+ (a_1a_{20} + 2a_2a_{18} - 4a_4a_{16} + 2a_6a_{14} + a_8a_{12} - 4a_1a_{19} + 2a_3a_{17} \\
&- 4a_5a_{15})\beta^{18}\delta^{10} + (a_1a_{22} + a_2a_{20} - 4a_4a_{18} + 4a_6a_{16} + a_8a_{14} + a_{10}a_{12} \\
&- 2a_1a_{21} + 4a_3a_{19} - 4a_{17}a_5 - 4a_7a_{15})\beta^{19}\delta^9 + (2a_1a_{24} + a_2a_{22} - 2a_4a_{20} \\
&+ 4a_6a_{18} + 2a_8a_{16} + a_{10}a_{14} + 760 - 2a_1a_{23} + 2a_3a_{21} - 8a_5a_{19} - 4a_7a_{19} \\
&- 2a_9a_{15})\beta^{20}\delta^8 + 2(-96 + a_2a_{24} - a_4a_{22} + a_6a_{20} + a_8a_{18} + a_{10}a_{16} + 380a_{14} \\
&+ 352 + 80a_1 + a_3a_{23} - 2a_5a_{21} - 4a_7a_{19} - a_9a_{17} - a_{11}a_{15})\beta^{21}\delta^7 + (-192 \\
&- 4a_4a_{24} + 2a_6a_{22} + a_8a_{20} + 2a_{10}a_{18} + 1520a_{16} + 704a_{14} - 160a_3 - 4a_5a_{23} \\
&- 4a_7a_{21} - 4a_9a_{19} - 2a_{11}a_{17} - 1992a_{15})\beta^{22}\delta^6 + (384a_4 + 4a_6a_{24} + a_8a_{22} \\
&+ a_{10}a_{20} + 1520a_{18} + 1408a_{16} + 320a_5 - 4a_7a_{23} - 2a_9a_{21} - 4a_{11}a_{19} \\
&- 1992a_{17} - 512a_{15})\beta^{23}\delta^5 + (-384a_6 + 2a_8a_{24} + a_{10}a_{22} + 760a_{20} \\
&+ 1408a_{18} + 320 - 2a_9a_{23} - 2a_{11}a_{21} - 3984a_{19} - 512a_{17})\beta^{24}\delta^4 \\
&+ (-a_8 + 2a_{10}a_{24} + 760a_{22} + 704a_{20} + 160a_9 - 2a_{11}a_{23} - 1992a_{21} \\
&- 1024a_{19})\beta^{25}\delta^3 + 8(-24a_{10} + 190a_{24} + 88a_{22} + 20a_{11} - 249a_2 \\
&- 64a_{21})\beta 26\delta^2 + 128(105 + 11a_{24} - 4a_{23})\beta^{27}\delta - 94208\beta^{28},
\end{aligned}
$$

$$
\begin{aligned}
S^{'} &= (2a_1a_{26} - a_1a_{25})\beta^{13}\delta^{13} + (2a_1a_{28} + 2a_2a_{26} - a_1a_{27} - a_3a_{25})\beta^{14}\delta^{12} \\
&+ (2a_1a_{30} + 2a_2a_{28} - 4a_4a_{26} - 2a_1a_{29} + a_3a_{27} - 2a_5a_{25})\beta^{15}\delta^{11} \\
&+ (6a_1a_{32} + 2a_2a_{30} - 4a_4a_{28} + 4a_6a_{26} - 2a_1a_{31} + 2a_3a_{29} - 2a_5a_{27} \\
&- 2a_7a_{25})\beta^{16}\delta^{10} + (2a_1a_{34} + 6a_2a_{32} - 4a_4a_{30} + 4a_6a_{28} + 2a_8a_{26} \\
&- a_1a_{33} + 2a_3a_{31} - 4a_5a_{29} - 2a_7a_{27} - a_9a\delta_{25})\beta^{17}\delta^9 + (6a_1a_{36} \\
&+ 2a_2a_{34} - 12a_4a_{32} + 4a_6a_{30} + 2a_8a_{28} + 2a_{10}a_{26} - a_1a_{35} + a_3a_{33} \\
&- 4a_5a_{31} - 4a_7a_{29} - a_9a_{27} - a_{11}a_{25})\beta^{18}\delta^8 + (-864a_1 + 6a_2a_{36} \\
&- 4a_4a_{34} + 12a_6a_{32} + 2a_8a_{30} + 2a_{10}a_{28} + 1520a_{26} + 1384 + a_3a_{35} \\
&- 2a_5a_{33} - 4a_7a_{31} - 2a_9a_{29} - a_{11}a_{27} - 996a_{25})\beta^{19}\delta^7 + (-864a_2 \\
&- 12a_{36}a_4 + 4a_6a_{34} + 6a_8a_{32} + 2a_{10}a_{30} + 1520a_{28} + 1408a_{26} - 1384 \\
&- 2a_5a_{35} - 2a_7a_{33} - 2a_9a_{31} - 2a_{11}a_{29} - 996 - 256a_{25})\beta^{20}\delta^6 \\
&+ (-4.432 + 12a_6a_{36} + 2a_8a_{34} + 6a_{10}a_{32} + 1520a_{30} + 1408a_{28} \\
&+ 2.1384 - 2a_7a_{35} - a_9a_{33} - 2a_{11}a_{31} - 1992a_{29} - 256a_{27})\beta^{21}\delta^5 \\
&+ (-4.432 + 6a_8a_{36} + 2a_{10}a_{34} + 3.1520a_{32} + 1408 + 2.1384 - a_9a_{35} \\
&- a_{11}a_{33} - 1992a_{31} - 512a_{29})\beta^{22}\delta^4 + (-864a_8 + 6a_{10}a_{36} + 1520a_{34} \\
&+ 3.1408 + 1384 - a_{11}a_{35} - 996a_{33} - 512a_{31})\beta^{23}\delta^3 \\
&+ (-864a_{10} + 1520.3a_{36} + 1408a_{34} + 1384a_{11} - 996a_{35} - 256a_{33})\beta^{24}\delta^2 \\
&+ (-432.1520 + 3.1408a_{36} + 996.1384 - 256a_{35})\beta^{25}\delta \\
&- (608256 - 354304)\beta^{26},
\end{aligned}
$$

Since only the term $94208\beta^{28}$ of $R^{'}r_{00} + \beta\delta S^{'}s_0$ in (3.15) seemingly does not contain δ, we must have $hp(1)V_1$ such that $r_{00} = \delta V_1$. We have $s_0 = 0$; $s_j = 0$, now (3.15) becomes

$$R^{'}r_{00} = 0, \tag{3.16}$$

which implies

$$r_{00} = 0;\ r_{ij} = 0\ and\ r_0 = 0;\ r_j = 0.$$

Conversely from $r_{00} = 0$, $r_0 = 0$ and $s_0 = 0$ we have $B^m_m = 0$. Thus the space with (3.1) is weakly-Berwald space. Thus we state that

Theorem 3.1. A Finsler space with the metric (3.1) is weakly Berwald space if and only if the following conditions holds;

(1) $\alpha^2 \neq 0(mod\beta)$ implies $r_{ij} = 0$ and $s_j = 0$.
(2) $\alpha^2 \equiv 0(mod\beta)$ implies $n = 2$, $b^2 = 0$ and $r_{ij} = 0$, $s_j = 0$ are satisfied, where $\alpha^2 = \beta\delta$, $\delta = d_i y^i$.

4 Conclusion

In this paper we investigate a Finsler space, where the (hv)- Ricci tensor G_{ij} vanishes, but the (hv)-curvature tensor G^h_{ijk} is not necessarily equal to zero. The aim of this paper to give an example for the so-called Weakly Berwald Finsler Space (WBFS), and a sufficient condition for the existence of a WBFS of second approximate Matsumoto type is determined also.

Competing Interests

The authors declare that they have no competing interests.

References

[1] Bacso S, Szilagyi B. On a weakly Berwald space of Kropina type. Math. Pannon. 2001;13(1):91-95.

[2] Matsumoto M. On a C-reducible finsler space. Tensor, N. S. 1972;24:29-37.

[3] Hashiguchi M, Ichijyo Y. On some special (α, β)-metrics. Rep. Fac. Sci. Kagoshima Univ. (Math., Phy., Chem.) 1975;8:39-46.

[4] Lee, Park Y, Ha-Young, Lee, Yound-duk. On a hypersurface of a special finsler space with a metric $\alpha + \frac{\beta^2}{\alpha}$. Korean J. Math, Sciences. 2001;8(1):93-101.

[5] Matsumoto M. Theory of finsler spaces with (α, β)-metric. Rep. Math. Phys. 1992;31:43-83.

[6] Shibata C. On finsler spaces with an (α, β)-metric. J. Hokkaido Univ. of Education. 1984;35:1-16.

[7] Lee IY, Lee MH. On weakly-berwald spaces of special (α, β)-metrics. Bull Korean Math. Soc. 2006;43(2):425-441.

[8] Bacso S, Yoshikawa R. Weakly-Berwald spaces. Publ. Math. Debrecen 2002;61(2):219-231.

[9] Yoshikawa R, Okubo K. The conditions for some (α, β)-metric spaces to be weakly-Berwald spaces. Proceedings of the 38-th Symposium on Finsler geometry. 2003;54-57.

[10] Shanker G, Yadav R. Weakly berwald space with first approximate matsumoto metric. Communicated in Journal of Tensor Society; 2012.

[11] Matsumoto M. Theory of finsler spaces with (α, β)-metric, Rep. Math. Phys. 1991; 30:15-20.

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