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On Numerical Treatments to Solve a Volterra - Hammerstein Integral Equation

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper a Volterra – Hammerstein integral equation (V-HIE), with two continuous kernels of position $k(x, y)$ and of time $F(t, \tau)$, is considered in the Banach space $C([0,1] \times [0, T]), T < 1$. The existence of a unique solution of the V-HIE, is discussed and proved. A quadratic numerical method is used to obtain a system of Hammerstein integral equations (SHIEs) in position and the existence of a unique solution of the SHIEs, under certain conditions, is proved. Moreover, we use two different methods, quadratic method (QM) and Simpson's rule (SR), to transform, in each method, the SHIEs into a nonlinear algebraic system (NAS). In addition, the existence of a unique solution of each algebraic system is guaranteed and proved. The Adomian decomposition method (ADM) is used to solve SHIEs without having to convert the system to a linearity. Finally, some applications contain numerical results, in some different time, are calculated and the error estimate, in each case, is computed.

Keywords: Volterra- Hammerstein Integral Equation (V-HIE); A System of Hammerstein Integral Equations (SHIEs); Quadratic Method (QM); Simpsosn's Rule (SR); Nonlinear Algebraic System (NAS); The Adomian Decomposition Method (ADM).

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1 Introduction

Integral equations play an important role in many subdivisions of linear and nonlinear functional analysis and their applications in potential theory, the theory of elasticity, mathematical physics, engineering, and electrostatics problems. In particular the mixed integral equations has many applications in different sciences, see [1-3]. Therefore, many different analytic and numeric methods are used to obtain the solution of the linear and nonlinear integral equations , In addition, different methods can be used to obtain the solution of the **NIEs** in the space $L_2[\Omega] \times C[0,T]$, where Ω is the domain of the contact problem considered in the position, while $t \in C[0,T], T < 1$ is the time. In fact, the numerical methods have played an important rule to obtain the numerical solution of the IEs in linear and nonlinear cases, see [4-12].

Consider the **V-HIE** of the second kind:

$$\mu\, u(x,t) = f(x,t) + \int_0^t \int_0^1 F(t,\tau)\, k\,(x,y)\, \gamma\big(y,\tau,u(y,\tau)\big) dy\, d\tau\ , \qquad \mu \neq 0\ . \tag{1}$$

Here, $f(x,t)$ and $\gamma\big(x,t,u(x,t)\big)$ are two given functions, while the function $u(x,t)$ is unknown in the space $C([0,1] \times [0,T])\ , T < 1$. The kernel of position $k\,(x,y)$ and the kernel of time $F(t,\tau)$ are continuous .The constant μ defines the kind of the integral equation.

In this paper, the existence of a unique solution of Eq. (1) is discussed and proved, using Banach fixed point theorem. A quadratic method is used to obtain a **SHIEs** in position, then the existence of a unique solution of this system is considered and proved. The quadratic method and Simpson's rule are applied to obtain a **NAS**. The existence of a unique solution of the produced **NAS** is guaranteed. Adomian method is applied on **SHIEs**, as another way, to solve this nonlinear system. Numerical examples are considered and the estimate error, in each case, is calculated, a comparison between these methods is served.

2 Existence of a Unique Solution of the Volterra-Hammerstein Integral Equation

In this section, we will prove the existence of a unique solution of (1), using Banach fixed point theorem. For this aim, we write **V-HIE** (1) in the integral operator form:

$$\bar{W}u(x,t) = \frac{1}{\mu}\, f(x,t) + Wu(x,t) \qquad , \quad \mu \neq 0 \quad . \tag{2}$$

$$Wu(x,t) = \frac{1}{\mu} \int_0^t \int_0^1 F(t,\tau)\, k\,(x,y)\, \gamma\big(y,\tau,u(y,\tau)\big) dy\, d\tau . \tag{3}$$

In addition, we assume the following conditions:

i) The kernel of position $k(x,y)$ is continuous, and satisfies

$|k(x,y)| \leq M$, (M is a constant) .

ii) The kernel of time $F(t,\tau)$ is a continuous function and satisfies

$|F(t,\tau)| \leq L$, (L is a constant) .

iii) The given function $f(x,t)$ with its partial derivatives with respect to position and time belong to $\mathrm{C}([0,1]\times[0,\mathrm{T}])$, and its norm is defined by:

$$\|f(x,t)\|_{C([0,1]\times[0,T])} = \max_{x,t}|f(x,t)| \leq H, (H \text{ is a constant })$$

iv) The known function $\gamma\big(x,t,u(x,t)\big)$ satisfies for the constants c_1, c_2 and $c = \max\{c_1, c_2\}$ the following conditions:

$$(a)\ \max_{x,t}\left|\gamma\big(x,t,u(x,t)\big)\right| \leq\ c_1\ \|u(x,t)\|_{C([0,1]\times[0,T])}\,.$$
$$(b)\left|\gamma\big(x,t,u(x,t)\big) - \gamma\big(x,t,\phi(x,t)\big)\right| \leq\ c_2\ |u(x,t)-\phi(x,t)|\,.$$

Theorem 1:

In view of the conditions (i-iv), **V-HIE** (1) has a unique solution in the Banach space $C([0,1]\times[0,T])$,under the condition $(T\ L\ M\ c) < |\mu|$.

Proof:

In view of the two formulas (2) and (3), we have

$$\|\overline{W}u(x,t)\| \leq \left|\frac{1}{\mu}\right| \|f(x,t)\| + \left|\frac{1}{\mu}\right| \left\|\int_0^t \int_0^1 |F(t,\tau)|\ |k\ (x,y)| \left|\gamma\big(y,\tau,u(y,\tau)\big)\right| dy\ d\tau\ \right\|.$$

Using the conditions (i) - (iv – a), we obtain

$$\|\overline{W}u(x,t)\| \leq \frac{H}{|\mu|} + \frac{\alpha}{|\mu|}\|u(x,t)\|\,, (\alpha =\ T\ L\ M\ c). \tag{4}$$

The previous inequality (4) shows that, the operator $\overline{W}$ maps the ball S_ρ into itself, where $\rho = \frac{H}{|\mu|-\alpha}$.

Since $\rho > 0$ & $H > 0$, therefore we have $\alpha < |\mu|$. Also, the inequality (4) shows that the operator W is bounded; moreover, the operator $\overline{W}$ is bounded.

For the two functions $u(x,t)$ and $\phi(x,t)$ in $C([0,1]\times[0,T])$, and from the formulas (2), (3), we find

$$\|\overline{W}u(x,t) - \overline{W}\phi(x,t)\| \leq \left|\frac{1}{\mu}\right| \left\|\int_0^t \int_0^1 |F(t,\tau)|\ |k\ (x,y)|\ \left|\gamma\big(y,\tau,u(y,\tau)\big) - \gamma\big(y,\tau,\phi(y,\tau)\big)\right|\ dy\ d\tau\ \right\|.$$

In view of the conditions (i), (ii) and (iv – b), we have

$$\|\overline{W}u(x,t) - \overline{W}\phi(x,t)\| \leq \frac{\alpha}{|\mu|}\ \|u(x,t)-\phi(x,t)\|. \tag{5}$$

From inequality (5), we see that if $\alpha < |\mu|$, then the operator $\overline{W}$ is continuous and contraction in the space $C([0,1]\times[0,T])$. So, from (4),(5) and the fixed point theorem, $\overline{W}$ has a unique fixed point which is the unique solution of (1).

3 A System of Hammerstein Integral Equations (SHIEs)

In this section, a **SHIEs** is obtained, using a quadratic numerical method, (Atkinson [13,14], Delves and Mohamed [15]). For this, we divide the interval $[0,T]$, $0 \le t \le T < 1$ into s subintervals, by means of the points: $0 = t_0 < t_1 < t_2 < \ldots < t_q < \cdots < t_p < \cdots < t_s = T$, where $t = t_p$, $\tau = t_q$, $p, q = 0,1,2, \ldots, s$, use the quadrature formula in the integral term of Eq. (1), then neglect the error, **V-HIE** (1) becomes

$$\mu u_p(x) - \sum_{q=0}^{p} w_q\, F_{p,q} \int_0^1 k(x,y)\, \gamma_q\left(y, u_q(y)\right) dy = f_p(x), \tag{6}$$

where we used the following notations:

$$u(x,t_p) = u_p(x), \qquad F(t_p,t_q) = F_{p,q} \;, \quad f(x,t_p) = f_p(x) \;, \gamma(y,t_q,\, u(y,t_q)) = \gamma_q\left(y, u_q(y)\right) \;,$$

and the weight function w_q satisfies , $w_q = h/2$; if $q = 0$, s ,and $w_q = h$; if $0 < q < s$. Eq. (6) represents a **SHIEs**, that can be solved using recurrence relations.

Now, let E be the set of all continuous functions $\mathrm{U}(x) = \{ u_0(x), u_1(x), \ldots, u_n, \ldots \}$, where $u_n(x)$ are functions in the space $C[0,1]$ and the norm in the space E is defined by: $\|\mathrm{U}\|_E = \max_n \max_x |u_n(x)|$. Then E is a Banach space.

4 The Existence of a Unique Solution of SHIEs

To discuss and prove the existence of a unique solution of the **SHIEs** (6) in the Banach space E, we write the (6) in the integral operator form

$$\bar{V} u_p(x) = \frac{1}{\mu} f_p(x) + V u_p(x) \,, \tag{7}$$

where

$$V u_p(x) = \frac{1}{\mu} \sum_{q=0}^{p} w_q\, F_{p,q} \int_0^1 k(x,y)\, \gamma_q\left(y, u_q(y)\right) dy \,. \tag{8}$$

In addition to condition (i) of theorem (1), we assume the following conditions:

(1) $\max_p \max_x \left| f_p(x) \right| = \| f(t) \|_E \le H^*$, (H^* is a constant) .

(2) $\sum_{q=0}^{s} \max_q \left| w_q\, F_{p,q} \right| \le L^*$, (L^* is a constant) .

(3) The known continuous functions $\gamma_P\left(x\,,\; u_p(x)\right), \forall p$ satisfies for the constants c_1^*, c_2^* and $c^* = \max\{ c_1^*, c_2^* \}$, the following conditions:

(a_1) $\max_p \max_x \left| \gamma_p\left(x, u_p(x)\right) \right| \le c_1^* \, \|\mathrm{U}(x)\|_E$.

(b_1) $\left| \gamma_p\left(x, u_p(x)\right) - \gamma_p\left(x, \phi_p(x)\right) \right| \le c_2^* \left| u_p(x) - \phi_p(x) \right|$.

Theorem 2:

The system (6) has a unique solution in the space E under the condition: $\alpha^* = L^*M\, c^* \ < |\mu|$.

Proof:

In view of the two formulas (7) and (8), we have

$$\left\|\bar{V} u_p(x)\right\| \leq \left|\frac{1}{\mu}\right| \left\|f_p(x)\right\| + \left|\frac{1}{\mu}\right| \sum_{q=0}^{p} \left|w_q\, F_{p,q}\right| \left\| \int_0^1 |k(x,y)| \left| \gamma_q\left(y, u_q(y)\right)\right| dy \right\|.$$

Using the conditions (i) and (1) - (3-a_1), we obtain

$$\left\|\bar{V} u_p(x)\right\| \leq \frac{H^*}{|\mu|} + \frac{\alpha^*}{|\mu|} \left\|u_p(x)\right\|, (\alpha^* = \ L^*M\, c^*\,). \tag{9}$$

The previous inequality (9) shows that, the operator $\bar{W}$ maps the ball S_θ into itself, where $\theta = \frac{H^*}{|\mu| - \alpha^*}$.

Since $\theta > 0$ and $H^* > 0$, therefore we have $\alpha^* < |\mu|$. Moreover, the inequality (9) involves that the operator $\bar{V}$ is bounded.

For the two functions $u_p(x)$ and $\phi_p(x)$ in E, and from the formulas (8), (9) we find

$$\left\|\bar{V} u_p(x) - \bar{V}\phi_p(x)\right\| \leq \left|\frac{1}{\mu}\right| \sum_{q=0}^{p} \left|w_q\, F_{p,q}\right| \left\| \int_0^1 |k(x,y)| \left| \gamma_q\left(y, u_q(y)\right) - \gamma_q\left(y, \phi_q(y)\right)\right| dy \right\|.$$

In view of the conditions (i), (2) with ($3 - b_1$), we have

$$\left\|\bar{V} u_p(x) - \bar{V}\phi_p(x)\right\| \leq \frac{\alpha^*}{|\mu|} \ \left\|u_p(x) - \phi_p(x)\right\|. \tag{10}$$

From inequality (10), we see that if $\alpha^* < |\mu|$, then the operator $\bar{V}$ is continuous and contraction in the space E. Consequently, from (9), (10) and fixed point theorem, $\bar{V}$ has a unique fixed point which is the unique solution of (6).

5 Numerical Methods

5.1 Quadratic numerical method

We use a quadratic numerical method, to transform the **SHIEs** (6) to a **NAS**, for this, we divide the interval [0,1] into N subintervals, by means of the points $0 = x_0 < x_1 < \cdots < x_N = 1$, where $x = x_i, i = 0,1,2,\ldots,N$, then use the quadrature formula in (6), we get

$$\mu\, u_{p,i} - \sum_{q=0}^{p} w_q\, F_{p,q} \sum_{j=0}^{N} v_j\, k_{i,j}\ \gamma_{q,j} = f_{p,i}\ , \tag{11}$$

where the notations: $u_{p,i} = u_p(x_i)$, $k_{i,j} = k\left(x_i, x_j\right)$, $\gamma_{q,j} = \gamma_q\left(x_j, u_q(x_j)\right)$, are used, and v_j are the weights of the numerical method $v_j = \begin{cases} h/2 & j = 0, m \\ h & 0 < j < m \end{cases}$.

5.2 Composite Simpson's rule

In this section, we approximate the **SHIEs** of (6), using Composite Simpson's rule, (Nadir and Rahmoune [16]) to obtain a **NAS**, in the form

$$\mu\, u_{p,i} = f_{p,i} + \frac{h}{3} \sum_{q=0}^{p} w_q\, F_{p,q} \left[4 \sum_{j=1}^{N/2} k_{i,2j-1}\ \gamma_{q,2j-1} + 2 \sum_{j=1}^{N-\frac{1}{2}} k_{i,2j}\ \gamma_{q,2j} + k_{i,0}\, \gamma_{q,0} + k_{i,N}\, \gamma_{q,N} \right], \tag{12}$$

which can adapted in the form

$$\mu\, u_{p,i} - \sum_{q=0}^{p} w_q\, F_{p,q} \sum_{j=0}^{N} v_j\, k_{i,j}\ \gamma_{q,j} = f_{p,i}\,, \tag{13}$$

where $v_j = 1/3$ if $j = 0, N$, $v_j = 2/3$ if j is even, and $v_j = 4/3$ if j is odd.

We can easily observe that the difference between the two algebraic systems (11) and (13) which are obtained by quadratic method and Simpson's rule, respectively, is in the values of the weights v_j.

6 The Existence of a Unique Solution of the (NAS)

In order to guarantee the existence of a unique solution of **NAS** in a Banach space l^∞, we write the **NAS** (11) or (13) in the integral operator form

$$\bar{Z}\, u_{p,i} = \frac{1}{\mu} f_{p,i} + Z\, u_{p,i}\ , \tag{14}$$

where

$$Z\, u_{p,i} = \frac{1}{\mu} \sum_{q=0}^{p} w_q\, F_{p,q} \sum_{j=0}^{N} v_j\, k_{i,j}\ \gamma_{q,j}\ . \tag{15}$$

Then, we assume in addition to condition (2), of theorem 2 the following conditions:

(a) $\sup\limits_{p,i} \left|f_{p,i}\right| \le H^{**}$ (H^{**} is a constant) .

(b) $\sup\limits_{i,N} \sum\limits_{j=0}^{N} \left|v_j\, k_{i,j}\right| \le E^*$, (E^* is a constant) .

(c) The known continuous functions $\gamma_{q,j}$ satisfy ($\forall\, q\, , j$) for the constants ξ_1 , ξ_2 and $\xi = \max\{\xi_1\, , \xi_2\}$ the following conditions:

$(\grave{a})$ $\sup\limits_{q,j} \left|\gamma_{q,j}(u_{q,j})\right| \le \xi_1 \left\|u_{q,j}\right\|_{l^\infty}$,

$(\grave{b})$ $\left|\gamma_{q,j}(u_{q,j}) - \gamma_{q,j}(\phi_{q,j})\right| \le \xi_2 \left|u_{q,j} - \phi_{q,j}\right|$.

Theorem 3:

Under the condition $\xi\ L^*\ E^* < |\mu|$, the **NAS** (11) or (13) has a unique solution in the Banach space l^∞.

Proof :

From the formulas (12) and (15), we obtain

$$\left|\bar{Z}\ u_{p,i}\right| = \left|\frac{1}{\mu}\right|\left[\left|f_{p,i}\right| + \sum_{q=0}^{p}\left|w_q\ F_{p,q}\right|\ \sum_{j=0}^{N}\left|v_j\ k_{i,j}\right|\left|\ \gamma_{q,j}\right|\right],$$

In view of the conditions ($\grave{1}$) - ($\grave{3}$-$\grave{\alpha}$), the above inequality takes the form

$$\left\|\bar{Z}\ u_{p,i}\right\| \leq \frac{H^{**}}{|\mu|} + \frac{\alpha^{**}}{|\mu|}\left\|u_{p,i}\right\| \quad , \qquad \alpha^{**} = \ \xi\ L^*\ E^*. \tag{16}$$

Inequality (16) shows that, the operator $\bar{V}$ maps the ball S_σ into itself, where

$$\sigma = \frac{H^{**}}{[|\mu| - \xi\ L^*\ E^*]} \quad . \tag{17}$$

Since $\sigma > 0$ and $H^{**} > 0$, therefore we have $\alpha^{**} < |\mu|$, moreover, the inequality (17) involves that the operator $\bar{Z}$ is bounded.

For the two functions $u_{p,i}$ and $\phi_{p,i}$ in l^∞, the formulas (14) and (15) yield

$$\left|\bar{Z}\ u_{p,i} - \bar{Z}\ \phi_{p,i}\right| \leq \frac{1}{|\mu|}\sum_{q=0}^{p}\left|w_q\ F_{p,q}\right|\ \sum_{j=0}^{N}\left|v_j\ k_{i,j}\right|\left|\gamma_{q,j}\left(u_{q,j}\right) - \gamma_{q,j}\left(\phi_{q,j}\right)\right|$$

Using the conditions $(2), (b)$ and $\left(c - \grave{b}\right)$, the above inequality takes the form

$$\left\|\bar{Z}\ u_{p,i} - \bar{Z}\ \phi_{p,i}\right\| \leq \frac{1}{|\mu|}\ \alpha^{**}\left\|u_{q,j} - \phi_{q,j}\right\| \ . \tag{18}$$

In view of inequality (18), we see that the operator $\bar{Z}$ is continuous in the Banach space l^∞. Moreover, $\bar{Z}$ is a contraction operator under the condition $\alpha^{**} < |\mu|$. From Banach space fixed point theorem, $\bar{Z}$ has a unique fixed point which is the unique solution of **NAS** (11) or (13).

Theorem 4:

If the sequence of continuous functions $\{\ f_j(x,t)\}$ converges uniformly to the function $f(x,t)$ in the space $C([0,1] \times [0,T])$, then under the conditions of theorem (1), the sequence $\{u_j(x,t)\}$ converges uniformly to the exact solution of Eq. (1) in $C([0,1] \times [0,T])$.

Proof:

The formula (1) with its approximate solution gives

$$\left|u(x,t) - u_j(x,t)\right| \leq \frac{1}{|\mu|}\left|f(x,t) - f_j(x,t)\right| + \frac{1}{|\mu|}\int_0^t\int_0^1 |F(t,\tau)|\ |k\ (x,y)|\left|\ \gamma\left(y,\tau,u(y,\tau)\right) - \ \gamma\left(y,\tau,u_j(y,\tau)\right)\right| dy\ d\tau$$

In view of the conditions of theorem 1, we get

$$\left\|u(x,t)-u_j(x,t)\right\| \le \frac{1}{[|\mu|-\;T\;L\;M\;c]}\left\|f(x,t)-f_j(x,t)\right\| \;. \tag{19}$$

Hence, $\left\|u(x,t)-u_j(x,t)\right\| \to 0$ since $\left\|f(x,t)-f_j(x,t)\right\| \to 0$ as $j \to \infty$.

Corollary 1:

The estimate total error define by the relation:

$$R_j = \left|\int_0^t\int_0^1 F(t,\tau)\;k\;(x,y)\;\gamma\big(y,\tau,u(y,\tau)\big)dy\;d\tau - \sum_{q=0}^{p} w_q\;F_{p,q}\;\sum_{j=0}^{N} v_j\;k_{i,j}\;\;\gamma_{q,j}\right|, \tag{20}$$

when $j = max\{N,p\} \to \infty$, the sums

$$\left\{\sum_{q=0}^{p} w_q\;F_{p,q}\;\sum_{j=0}^{N} v_j\;k_{i,j}\;\;\gamma_{q,j}\right\} \to \left\{\int_0^t\int_0^1 F(t,\tau)\;k\;(x,y)\;\gamma\big(y,\tau,u(y,\tau)\big)dy\;d\tau\right\},$$

and the solution of the **NAS** (11) or (13) becomes the solution of **V-HIE** (1) .

In view of theorem 4, we can deduce that the total error R_j satisfies $\lim_{j\to\infty} R_j = 0$.

7 The Adomian Decomposition Method (ADM)

ADM is a semi-analytical method for solving integral equations, ordinary or partial differential equations, algebraic equations, and so on, see [17-19]. The ADM involves separating the equation into linear and nonlinear portions. The nonlinear portion is decomposed into a series of Adomian polynomials, and the solution is generated in the form of a series whose terms are determined by a recursive relationship using these Adomian polynomials, so, the solution can be determined by calculation of the Adomian polynomials which allow for solution convergence of the nonlinear portion of the equation, without simply linearizing the system. In the references [20-23] the convergence of ADM are discussed and proved by different methods.

Consider the following system of integral equations:

$$\mu u_p(x) - \sum_{q=0}^{p} w_q\;F_{p,q}\int_0^1 k(x,y)\;\gamma_q\left(y,u_q(y)\right)dy = f_p(x). \tag{21}$$

Assume the functions $u_p(x)$ can be written as an infinite series:

$$u_p(x) = \sum_{i=0}^{\infty} u_{p,i}(x). \tag{22}$$

While the nonlinear term $\gamma_p\left(x,u_p(x)\right)$ of (21) is decomposed into an infinite series

$$\gamma_p\left(x,u_p(x)\right) = \sum_{i=0}^{\infty} A_{p,i}\;, \tag{23}$$

where the traditional formula of $A_{k,n}$ is:

$$A_{p,i} = \frac{1}{i\,!}\left(\frac{d^i}{d\eta^i}\ \gamma_p\left(\sum_{n=0}^{\infty}\eta^n\,u_{p,n}\right)\right)_{\eta=0}, \tag{24}$$

after applying the ADM on equation (21), we have

$$u_{p,0}(x) = \frac{1}{\mu}\ f_p(x)\,; \qquad u_{p,i}(x) = \frac{1}{\mu}\sum_{q=0}^{p} w_q\,F_{p,q}\int_0^1 k(x,y)\,A_{p,(i-1)}(y)\,dy\,,\ (\ i \geq 1). \tag{25}$$

Where the so-called Adomian polynomials $A_{p,i}$ can be evaluated for the nonlinear function $\gamma_p\left(x, u_p(x)\right)$, therefore the Adomian polynomials are given by

$$\begin{aligned}
A_{p,0} &= \gamma_p(u_{p,0})\,, \\
A_{p,1} &= u_{p,1}\ \grave{\gamma}_p(u_{p,0})\,, \\
A_{p,2} &= \frac{1}{2}u_{p,1}^2\ \grave{\grave{\gamma}}_p(u_{p,0}) + u_{p,2}\ \grave{\gamma}_p(u_{p,0})\,, \\
A_{p,3} &= \frac{1}{6}u_{p,1}^3\ \gamma_p^{(3)}(u_{p,0}) + u_{p,1}u_{p,2}\ \grave{\grave{\gamma}}_p(u_{p,0}) + u_{p,3}\ \grave{\gamma}_p(u_{p,0})\,. \\
A_{p,4} &= \frac{1}{24}u_{p,1}^4\ \gamma_p^{(4)}(u_{p,0}) + \frac{1}{2}u_{p,1}^2 u_{p,2}\ \gamma_p^{(3)}(u_{p,0}) + \left(\frac{1}{2}u_{p,2}^2 + u_{p,1}u_{p,3}\right)\grave{\grave{\gamma}}_p(u_{p,0}) \\
&\qquad + u_{p,4}\ \grave{\gamma}_p(u_{p,0}).
\end{aligned} \tag{26}$$

The determination of $u_{p,0}$ and $u_{p,1}$ leads to the determination of $A_{p,1}$ that will allows us to determine $u_{p,2}$, and so on. This in turn will lead to the complete determination of the components of $u_{p,i}$,$i \geq 1$, upon using the second part of (25). The series solution follows immediately after using (22). The obtained series solution converge to an exact solution of **V-HIE** (1).

8 Numerical Results

Consider the **V-HIE** (1), at $\mu = 1$, and the times $T = 0.008, T = 0.07$ and $T = 0.5$, then divided the position interval by $N = 100$ units, and the time interval in $s = 4$. The next tables give us an exact solution with the Simpson's approximate solution (App. SR), quadratic approximate solution (App. QD), and Adomian decomposition method (App. ADM), with their corresponding errors (E. QD.), (E. SR.), (E. ADM), respectively, the diagrams explain the difference between these results.

Application 1:

Consider the integral equation:

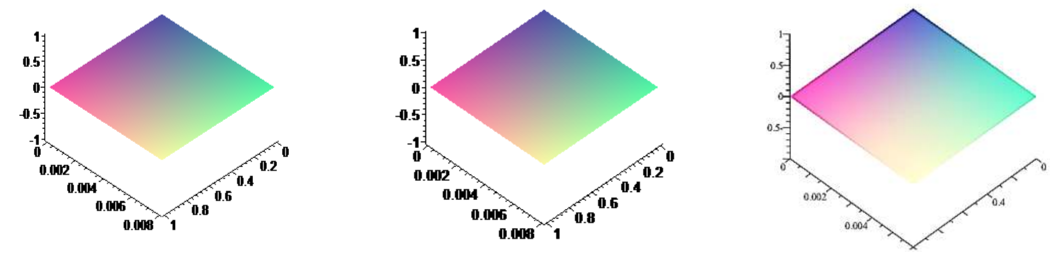
$$u(x,t) = f(x,t) + \int_0^t\int_0^1 t\,\tau\ \frac{(x+y)}{(3-x)}\ \left(u(y,\tau)\right)^2 dy\,d\tau\,, \tag{27}$$

The exact solution : $u(x,t) = x^2\ t^3$.

Tables 1, 2 and 3 describe the errors of the approximate solution, of equation (27), by SM and the QM, their accuracies are 10^{-9}, $10^{-6}, 10^{-3}$, whilst, the accuracies of ADM are $10^{-9}, 10^{-7}$, 10^{-4} , when $T = 0.008, 0.07$, $T = 0.5$, respectively.

Table 1. Case 1: $N = 100$, $T = 0.008$

X	Exact	App. SR	E. SR	App. QD	E. QD	App. ADM	E. ADM
0	0	0	0	0	0	0	0
0.1	5.1200E-09	5.1200E-09	0	5.1200E-09	0	5.1200E-09	0
0.2	2.0480E-08	2.0480E-08	0	2.0480E-08	0	2.0480E-08	0
0.3	4.6080E-08	4.6080E-08	0	4.6080E-08	0	4.6080E-08	0
0.4	8.1920E-08	8.1920E-08	0	8.1920E-08	0	8.1920E-08	0
0.5	1.2800E-07	1.2800E-07	0	1.2800E-07	0	1.2800E-07	0
0.6	1.8432E-07	1.8432E-07	0	1.8432E-07	0	1.8432E-07	0
0.7	2.5088E-07	2.5088E-07	0	2.5088E-07	0	2.5088E-07	0
0.8	3.2768E-07	3.2768E-07	0	3.2768E-07	0	3.2768E-07	0
0.9	4.1472E-07	4.1472E-07	0	4.1472E-07	0	4.1472E-07	0
1	5.1200E-07	5.1200E-07	0	5.1200E-07	0	5.1200E-07	0

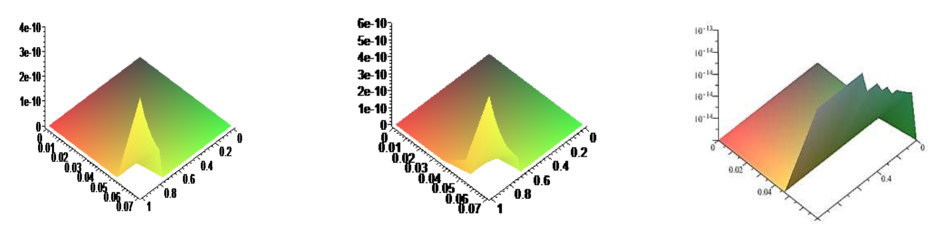


Error of Simpson's App. Sol. **Error of Quadratic's App. Sol.** **Error of Adomian App. Sol.**

Fig. 1.

Table 2. Case 2: $\mathrm{N} = 100$, $\mathrm{T} = 0.07$

X	Exact	App. SR	E. SR	App. QD	E. QD	App. ADM	E. ADM
0	0	0	0	0	0	0	5.0E-14
0.1	3.430E-06	3.430E-06	0	3.430E-06	0	3.430E-06	6.0E-14
0.2	1.372E-05	1.372E-05	0	1.372E-05	0	1.372E-05	7.0E-14
0.3	3.087E-05	3.087E-05	0	3.087E-05	0	3.087E-05	8.0E-14
0.4	5.488E-05	5.488E-05	0	5.488E-05	0	5.488E-05	8.0E-14
0.5	8.575E-05	8.575E-05	0	8.575E-05	0	8.575E-05	1.0E-13
0.6	1.235E-04	1.235E-04	0	1.235E-04	0	1.235E-04	1.0E-13
0.7	1.681E-04	1.681E-04	0	1.681E-04	1.06E-10	1.681E-04	1.0E-13
0.8	2.195E-04	2.195E-04	1.33E-10	2.195E-04	1.99E-10	2.195E-04	1.0E-13
0.9	2.778E-04	2.778E-04	2.35E-10	2.778E-04	3.51E-10	2.778E-04	1.0E-13
1	3.430E-04	3.430E-04	3.94E-10	3.430E-04	5.90E-10	3.430E-04	5.0E-13

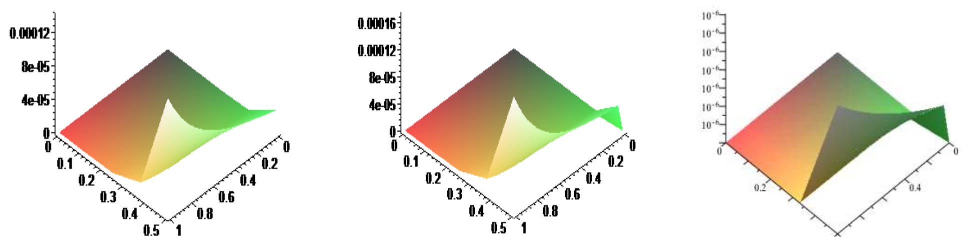


Error of Simpson's App. Sol. **Error of Quadratic's App. Sol.** **Error of Adomian App. Sol.**

Fig. 2.

Table 3. Case 3: $\mathbf{N = 100}$, $\mathbf{T = 0.5}$

X	Exact	App. SR	E. SR	App. QD	E. QD	App. ADM	E. ADM
0	0	0	0	0	0	0	0
0.1	1.250E-03	1.281E-03	3.143E-05	1.293E-03	4.308E-05	1.252E-03	2.472E-06
0.2	5.000E-03	5.036E-03	3.608E-05	5.046E-03	4.598E-05	5.003E-03	2.834E-06
0.3	1.125E-02	1.129E-02	4.119E-05	1.130E-02	4.959E-05	1.125E-02	3.224E-06
0.4	2.000E-02	2.005E-02	4.705E-05	2.005E-02	5.429E-05	2.000E-02	3.643E-06
0.5	3.125E-02	3.130E-02	5.404E-05	3.131E-02	6.072E-05	3.125E-02	4.096E-06
0.6	4.500E-02	4.506E-02	6.284E-05	4.507E-02	6.984E-05	4.500E-02	4.587E-06
0.7	6.125E-02	6.132E-02	7.437E-05	6.133E-02	8.308E-05	6.126E-02	5.120E-06
0.8	8.000E-02	8.009E-02	8.999E-05	8.010E-02	1.025E-04	8.001E-02	5.702E-06
0.9	0.10125	0.101362	1.116E-04	0.101381	1.308E-04	1.013E-01	6.339E-06
1	0.125	0.125142	1.417E-04	0.125172	1.719E-04	1.250E-01	7.040E-06



Error of Simpson's App. Sol. **Error of Quadratic's App. Sol.** **Error of Adomian App. Sol.**

Fig. 3.

Application 2:

Consider the integral equation:

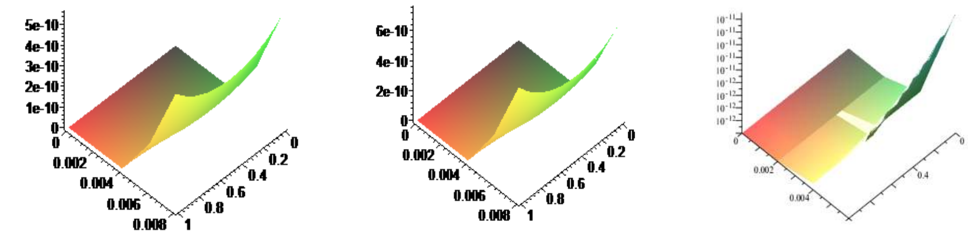
$$u(x,t) = f(x,t) + \int_0^t \int_0^1 e^{t-\tau} \; \frac{y}{x+1} \left(u(y,\tau)\right)^3 dy\, d\tau. \tag{28}$$

The exact solution : $u(x,t) = t\cosh(x)/2$.

Tables 4, 5 and 6 describe the errors of the approximate solution, of equation (28), by SM and QM, the accuracies are 10^{-7}, 10^{-4}, 10^{-2} , the accuracies of ADM are $10^{-8}, 10^{-5}$, 10^{-3} , when $T =$ 0.008, 0.07 and $T = 0.5$, respectively.

Table 4. Case 4: $\mathbf{N = 100}$, $\mathbf{T = 0.008}$

X	Exact	App. SR	E. SR	App. QD	E. QD	App. ADM	E. ADM
0	4.000E-03	4.000E-03	5.54E-10	4.000E-03	6.61E-10	4.000E-03	7.60E-11
0.1	4.020E-03	4.020E-03	5.06E-10	4.020E-03	6.04E-10	4.020E-03	6.90E-11
0.2	4.080E-03	4.080E-03	4.73E-10	4.080E-03	5.67E-10	4.080E-03	6.40E-11
0.3	4.181E-03	4.181E-03	4.49E-10	4.181E-03	5.43E-10	4.181E-03	5.90E-11
0.4	4.324E-03	4.324E-03	4.36E-10	4.324E-03	5.32E-10	4.324E-03	5.50E-11
0.5	4.511E-03	4.511E-03	4.31E-10	4.511E-03	5.33E-10	4.511E-03	5.10E-11
0.6	4.742E-03	4.742E-03	4.35E-10	4.742E-03	5.46E-10	4.742E-03	4.80E-11
0.7	5.021E-03	5.021E-03	4.48E-10	5.021E-03	5.72E-10	5.021E-03	4.50E-11
0.8	5.349E-03	5.349E-03	4.73E-10	5.349E-03	6.14E-10	5.350E-03	4.20E-11
0.9	5.732E-03	5.732E-03	5.09E-10	5.732E-03	6.75E-10	5.732E-03	4.00E-11
1	6.172E-03	6.172E-03	5.62E-10	6.172E-03	7.58E-10	6.172E-03	3.80E-11

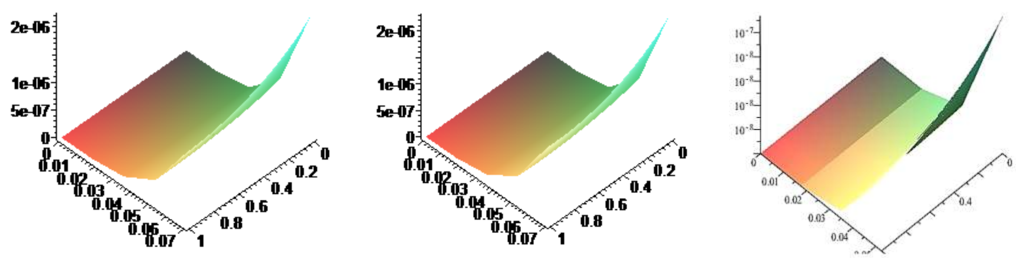


Error of Simpson's App. Sol. **Error of Quadratic's App. Sol.** **Error of Adomian App. Sol.**

Fig. 4.

Table 5. Case 5: $N = 100,\ T = 0.07$

X	Exact	App. SR	E. SR	App. QD	E. QD	App. ADM	E. ADM
0	3.500E-02	3.500E-02	2.228E-06	3.500E-02	2.300E-06	3.500E-02	4.56E-07
0.1	3.518E-02	3.518E-02	2.028E-06	3.518E-02	2.094E-06	3.517E-02	4.15E-07
0.2	3.570E-02	3.570E-02	1.864E-06	3.570E-02	1.927E-06	3.570E-02	3.80E-07
0.3	3.659E-02	3.659E-02	1.730E-06	3.659E-02	1.792E-06	3.659E-02	3.51E-07
0.4	3.784E-02	3.784E-02	1.618E-06	3.784E-02	1.683E-06	3.784E-02	3.26E-07
0.5	3.947E-02	3.947E-02	1.527E-06	3.947E-02	1.595E-06	3.947E-02	3.04E-07
0.6	4.149E-02	4.149E-02	1.452E-06	4.149E-02	1.527E-06	4.149E-02	2.85E-07
0.7	4.393E-02	4.393E-02	1.393E-06	4.393E-02	1.476E-06	4.393E-02	2.68E-07
0.8	4.681E-02	4.681E-02	1.348E-06	4.681E-02	1.443E-06	4.681E-02	2.53E-07
0.9	5.016E-02	5.016E-02	1.319E-06	5.016E-02	1.430E-06	5.016E-02	2.40E-07
1	5.401E-02	5.401E-02	1.305E-06	5.401E-02	1.437E-06	5.401E-02	2.28E-07

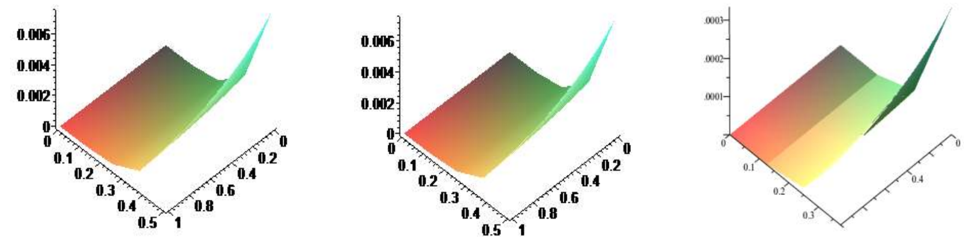


Error of Simpson's App. Sol. **Error of Quadratic's App. Sol.** **Error of Adomian App. Sol.**

Fig. 5

Table 6. Case 6: $N = 100,\ T = 0.5$

X	Exact	App. SR	E. SR	App. QD	E. QD	App. ADM	E. ADM
0	0.25	0.257463	0.007463	0.257492	0.007492	0.248635	0.001365
0.1	0.251251	0.258036	0.006785	0.258062	0.006811	0.25001	0.001241
0.2	0.255017	0.261238	0.006222	0.261263	0.006246	0.253879	0.001138
0.3	0.261335	0.267081	0.005746	0.267105	0.005771	0.260284	0.00105
0.4	0.270268	0.275608	0.005340	0.275633	0.005365	0.269293	0.000975
0.5	0.281906	0.286897	0.004990	0.286923	0.005016	0.280996	0.00091
0.6	0.296366	0.301052	0.004686	0.301081	0.004714	0.295513	0.000853
0.7	0.313792	0.318212	0.00442	0.318244	0.004452	0.312989	0.000803
0.8	0.334359	0.338545	0.004187	0.338581	0.004223	0.3336	0.000758
0.9	0.358272	0.362253	0.003982	0.362295	0.004023	0.357553	0.000719
1	0.385771	0.389572	0.003802	0.389621	0.003851	0.385088	0.000683



Error of Simpson's App. Sol. **Error of Quadratic's App. Sol.** **Error of Adomian App. Sol.**

Fig. 6.

9 Conclusion

- A **SHIEs** in position was formulated from a **V-HIE** by using quadratic numerical method.
- Using Banach Fixed Point theorem, the existence of a unique solution of the V-HIE, was discussed and proved in the space $L_2[0,1] \times C[0,T]$, $T < 1$.
- The existence of a unique solution of the **SHIEs**, under certain condition, was discussed.
- A **NAS** was configured from **SHIEs** by using two different methods, QM and SR.
- ADM was used as another way to process a **SHIEs** by a convergent series, without be compelled to convert it to a linear system.
- Some applications contain numerical results, in different times, were calculated.
- This equation can be solved also by using many methods as Homotopy, Galerkin, and Collocation.
- We can consider this equation in two dimension in the position, with a singular kernel.
- We observed from the numerical results of the studied cases that:

1. Time playing a great role in the results, where the error values was increasing according to the increasing of time.
2. In all the calculated results, **SR** can get a bit more accuracy approximate solution than **QM**, but the most accurate is **ADM**.
3. The approximate solutions are more accurate when the exact solution is a polynomial, as in application 1.
4. The error values, approximately, was vanished at the initial times (when $T < 0.01$) , for all $x \in [0,1]$.

Competing Interests

Author has declared that no competing interests exist.

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