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The effect of time-varying delay damping on the stability of porous elastic system

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Abstract: In the present work, we study the effect of time varying delay damping on the stability of a one-dimensional porous-viscoelastic system. We also illustrate our findings with some examples. The present work improve and generalize existing results in the literature.

Keywords: Optimal decay; Porous system; Delay; Viscoelasticity; Convexity.

MSC: 35B35; 35D35; 35B40; 93D20.

1. Introduction

Problems in porous media are essential in the field of petroleum engineering, soil mechanics, power technology, biology, material science, etc. Thus, this has attracted the attention of scientists and mathematicians in particular, see for instance the results in [1–8] and the references cited therein for related theory of porous-elastic materials. The basic equations describing the motion of a classical porous system are given by

$$\begin{cases} \rho\varphi_{tt} - S_x = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ J\psi_{tt} - G_x - Q = 0, & \text{in } (0,L)\times\mathbb{R}_+, \end{cases} \tag{1}$$

where $\varphi = \varphi(x,t)$ and $\psi = \psi(x,t)$ are the displacements of solid elastic material and the volume fraction, respectively. The physical parameters ρ and J are respectively, mass density and product of the equilibrated inertia by the mass density. The constitutive laws S, G and Q are: stress tensor, equilibrated stress vector and equilibrated body force, respectively. Time delays occur in systems modeling different types of phenomena in areas such as: biosciences, medicine, physics, chemical and structural engineering. These phenomena depend naturally on the present state and past history of the system. It is a well known fact that the presence of a delay term in a system, which is a priori a stable system, might cause an instability in the system, see for instance, the result of Nicaise and Pignotti [9]. In past decades, a great number of researchers have investigated the effect of delay on the stability of various systems or wave equations (with or without memory), see for example [10–15] and references therein. Back to system (1), we should mention that, there are very few results in literature that studied the effect of delay on this system. With memory and time varying delay dampings, the constitutive laws in (1) are given by

$$\begin{cases} S = k\varphi_x + b\psi, \\ G = \delta\psi_x - \displaystyle\int_0^t g(t-s)\psi_x(.,s)ds, \\ Q = -b\varphi_x - a\psi - \mu_1\psi_t - \mu_2\psi_t(.,t-\tau(t)), \end{cases} \tag{2}$$

where the constitutive physical parameters, k, b, δ, a satisfies

$$k > 0,\ \delta > 0,\ a > 0,\ b^2 < ka, \tag{3}$$

μ_1, μ_2 are real constants, $\tau(t) > 0$ is the time-dependent delay and g is a given function to be specified later. For simplicity, we set $L = 1$, then substituting (2) into (1), we arrive at the following porous-viscoelastic system with varying time dependent delay;

$$
\begin{cases}
\rho\varphi_{tt} - k\varphi_{xx} - b\psi_x = 0, & \text{in } (0,1)\times\mathbb{R}_+,\\
J\psi_{tt} - \delta\psi_{xx} + b\varphi_x + a\psi + \int_0^t g(t-s)\psi_{xx}(x,s)ds + \mu_1\psi_t + \mu_2\psi_t(.,t-\tau(t)) = 0, & \text{in } (0,1)\times\mathbb{R}_+,\\
\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = 0, & t\in\mathbb{R}_+,\\
\varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x),\ x\in(0,1), & \\
\psi_t(x,t) = f_0(x,t), \quad \text{in } (0,1)\times(-\tau(0),0). &
\end{cases} \tag{4}
$$

When $g = \mu_1 = \mu_2 = 0$, Quintanilla [16] investigated

$$
\begin{cases}
\rho\varphi_{tt} - k\varphi_{xx} - b\psi_x = 0, & \text{in } (0,1)\times(0,+\infty),\\
J\psi_{tt} - \delta\psi_{xx} + b\varphi_x + a\psi - \gamma\psi_t = 0, & \text{in } (0,1)\times(0,+\infty),
\end{cases}
$$

where $\mu_1 = \gamma > 0$ and showed the lack of exponential stability. However, he established a slow non-exponential decay result. Casas and Quintanilla [17] studied

$$
\begin{cases}
\rho\varphi_{tt} - k\varphi_{xx} - b\psi_x + \beta\theta_x = 0, & \text{in } (0,1)\times(0,+\infty),\\
J\psi_{tt} - \delta\psi_{xx} + b\varphi_x + a\psi - m\theta + \gamma\psi_t = 0, & \text{in } (0,1)\times(0,+\infty),\\
c\theta_t - \kappa\theta_{xx} + \beta\varphi_{xt} + m\psi_{xt} = 0, & \text{in } (0,1)\times(0,+\infty),
\end{cases} \tag{5}
$$

and improved the result in [16] (exponential stability). Soufyane *et al.*, [18] considered (5) with viscoelastic damping on the boundaries and proved a general decay estimate. Recently, Apalara [19] looked at

$$
\begin{cases}
\rho\varphi_{tt} - k\varphi_{xx} - b\psi_x = 0, & \text{in } (0,1)\times(0,+\infty),\\
J\psi_{tt} - \delta\psi_{xx} + b\varphi_x + a\psi + \int_0^t g(t-s)\psi_{xx}(x,s)ds = 0, & \text{in } (0,1)\times(0,+\infty),
\end{cases}
$$

where the memory g satisfies $g'(t) \le -\xi(t)g(t)$ and established a general decay estimate. Feng and Apalara [20] improved the result in [19] when the relaxation function g satisfies $g'(t) \le -\xi(t)H(g(t))$.

For results in porous systems with delay damping, not much has been done in this direction. we refer the reader to the result of Khochemane *et al.*, [21], where they considered a porous elastic system with weak internal damping and constant delay damping. Precisely, they studied

$$
\begin{cases}
\rho\varphi_{tt} - k\varphi_{xx} - b\psi_x = 0, & \text{in } (0,1)\times(0,+\infty),\\
J\psi_{tt} - \delta\psi_{xx} + b\varphi_x + a\psi + \alpha(t)g(\psi_t) + \mu_1\psi_t + \mu_2\psi_t(.,t-\tau) = 0, & \text{in } (0,1)\times(0,+\infty),
\end{cases} \tag{6}
$$

and proved a general decay result provided g satisfies

$$
\begin{cases}
c_1|s| \le |g(s)| \le c_2|s|, & \text{if } |s| \ge \epsilon,\\
s^2 + g^2(s) \le G^{-1}(sg(s)), & \text{if } |s| \le \epsilon.
\end{cases}
$$

Recently, Borges Filho and Santos [22] considered

$$
\begin{cases}
\rho\varphi_{tt} - k\varphi_{xx} - b\psi_x = 0, & \text{in } (0,1)\times(0,+\infty),\\
J\psi_{tt} - \delta\psi_{xx} + b\varphi_x + a\psi + \mu_1\psi_t + \mu_2\psi_t(.,t-\tau(t)) = 0, & \text{in } (0,1)\times(0,+\infty),
\end{cases}
$$

and showed that the system is exponentially stable. More related results can be found in [23–27] and references therein.

The novelty of this work is to study the stability of system (7). In fact, we show that the solution energy has an optimal decay estimate even in the presence of time varying delay term, from which the results in [21,22] are particular cases. To the best of our knowledge, system (7) has not been considered before in the literature. The rest of work is organized as follows: In Section 2, we recall some preliminaries and assumptions on the memory term. In Section 3, we state and prove several lemmas needed for establishing our main results. In Section 4, we establish the uniform stability result and in Section 5, we give some examples in support of our results.

2. Problem setting and assumptions

In this work, we consider the following system:

$$\begin{cases} \rho\varphi_{tt} - k\varphi_{xx} - b\psi_x = 0, & \text{in } (0,1)\times\mathbb{R}_+, \\ J\psi_{tt} - \delta\psi_{xx} + b\varphi_x + a\psi + \int_0^t g(t-s)\psi_{xx}(x,s)ds + \mu_1\psi_t + \mu_2\psi_t(.,t-\tau(t)) = 0, & \text{in } (0,1)\times\mathbb{R}_+, \\ \varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = 0, & t\in\mathbb{R}_+, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), & x\in(0,1), \\ \psi_t(x,t) = f_0(x,t), \quad \text{in } (0,1)\times(-\tau(0),0). \end{cases} \tag{7}$$

In addition to (3), we need the following:

Assumptions

(A1) The relaxation function $g : [0,+\infty) \to (0,+\infty)$ is a $\mathcal{C}^1$ non-increasing function and satisfies

$$g(0) > 0, \qquad \delta - \int_0^{+\infty} g(s)ds = l > 0. \tag{8}$$

(A2) There exists a $\mathcal{C}^1-$function $M : [0,+\infty) \to [0,+\infty)$, which is either linear or is a strictly increasing and strictly convex $\mathcal{C}^2$ function on $[0,\alpha]$, $\alpha > 0$, $\alpha \le g(0)$, with $M(0) = M'(0) = 0$, such that

$$g'(t) \le -\xi(t)M(g(t)), \quad \forall\, t \ge 0, \tag{9}$$

where ξ is a positive non-increasing differentiable function.

(A3) There exist $\tau_0, \tau_1 > 0$ such that

$$0 < \tau_0 \le \tau(t) \le \tau_1,\ \forall\, t > 0. \tag{10}$$

(A4)

$$\tau(t) \in W^{2,+\infty}(0,T) \text{ and } \tau'(t) \le d < 1,\ \forall\, t,T > 0. \tag{11}$$

(A5) The real constants μ_1 and μ_2 satisfies $|\mu_2| < \mu_1\sqrt{1-d}$.

From $(A1)$ and $(A2)$, we can deduce the following:

(I) It follows from $(A1)$ that $\lim_{t\to\infty} g(t) = 0$. Thus, there exists $t_0 \ge 0$ large enough, such that

$$g(t_0) = \alpha \quad \text{and} \quad g(t) \le \alpha, \quad \forall\, t \ge t_0. \tag{12}$$

(II) Since g and ξ are positive, non-increasing and continuous functions, in addition to M being a positive continuous function, it follows that, for all $t \in [0,t_0]$,

$$\left.\begin{array}{l} 0 < g(t_0) \le g(t) \le g(0) \\ 0 < \xi(t_0) \le \xi(t) \le \xi(0) \end{array}\right\} \Rightarrow \beta_1 \le \xi(t)M(g(t)) \le \beta_2$$

for some positive constants β_1 and β_2. Hence,

$$g'(t) \le -\xi(t)M(g(t)) \le -\frac{\beta_1}{g(0)}g(0) \le -\frac{\beta_1}{g(0)}g(t),\ \forall\, t \in [0,t_0]. \tag{13}$$

(III) M has an extension $\overline{M}$, which is a strictly increasing and strictly convex $\mathcal{C}^2$ function on $(0,\infty)$. As an example, given that $M(\alpha) = a_1$, $M'(\alpha) = a_2$ and $M''(\alpha) = a_3$, then we can define $\overline{M}$ by

$$\overline{M}(t) = \frac{a_3}{2}t^2 + (a_2 - a_3\alpha)t + \left(a_1 + \frac{a_3}{2}\alpha^2 - a_2\alpha\right),\ \forall\, t > \alpha. \tag{14}$$

From now on, C denotes a positive constant that may change within lines or from line to line. We denote by $\|.\|_2$ the usual norm in $L^2(0,1)$ and define the following spaces:

$$L^2_*(0,1) = \left\{ w \in L^2(0,1) : \int_0^1 w(x)dx = 0 \right\},\ H^1_*(0,1) = H^1(0,1) \cap L^2_*(0,1),$$

and

$$H_*^2(0,1) = H^2(0,1) \cap H_*^1(0,1).$$

Let $W = (\varphi, \varphi_t, \psi, \psi_t)$, $W_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1)$ and set $\mathcal{H} = H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1)$, $\mathcal{H}_1 = H_*^2(0,1) \times H_*^1(0,1) \times H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1)$. We have the following well-possedness result, which is obtained by using the Classical Faedo-Galerkin method.

Theorem 1. *Suppose assumptions $(A1) - (A5)$ hold. Let $W_0 \in \mathcal{H}$ and $f_0 \in H^1((0,1) \times (-\tau(0),0))$, then (7) possesses a unique weak solution $W \in \mathcal{C}([0,+\infty), \mathcal{H})$. Moreover, if $W_0 \in \mathcal{H}_1$ and $f_0 \in H^2((0,1) \times (-\tau(0),0))$, then the solution is more regular in the class $W \in \mathcal{C}([0,+\infty), \mathcal{H}_1) \cap \mathcal{C}^1([0,+\infty), \mathcal{H})$.*

We recall the following useful lemmas that will be applied repeatedly throughout this article.

Lemma 1. *Let $w \in L^2_{loc}([0,+\infty), L^2(0,1))$, we have*

$$\int_0^1 \left(\int_0^t g(t-s)(w(t) - w(s))ds \right)^2 dx \leq (1-l)(g \circ w)(t), \tag{15}$$

and

$$\int_0^1 \left(\int_0^x w(y,t)dy \right)^2 dx \leq \|w(t)\|_2^2, \tag{16}$$

where $(g \circ w)(t) = \int_0^t g(t-s)\|w(t) - w(s)\|_2^2 ds$.

Lemma 2. *Let $w \in H_0^1(0,1)$, then*

$$\int_0^1 \left(\int_0^t g(t-s)(w(t) - w(s))ds \right)^2 dx \leq C_p(1-l)(g \circ w)(t), \tag{17}$$

where $C_p > 0$ is the Poincaré's constant and $(g \circ w)(t) = \int_0^t g(t-s)\|w(t) - w(s)\|_2^2 ds$.

Lemma 3. *Let (φ, ψ) be the solution of (7). Then, for any $0 < \alpha < 1$, we have*

$$\int_0^1 \left(\int_0^t g(t-s) \left(\psi_x(s) - \psi_x(t)\right) ds \right)^2 dx \leq A_\alpha \left(h \circ \psi_x\right)(t), \tag{18}$$

where $h(t) = \alpha g(t) - g'(t)$ and $A_\alpha = \int_0^{+\infty} \frac{g^2(s)}{\alpha g(s) - g'(s)} ds$.

Proof. Cauchy-Schwarz inequality gives

$$\begin{aligned}
\int_0^1 \left(\int_0^t g(t-s) \left(\psi_x(s) - \psi_x(t)\right) ds \right)^2 dx &= \int_0^1 \left(\int_0^t \frac{g(t-s)}{\sqrt{h(t-s)}} \sqrt{h(t-s)} \left(\psi_x(s) - \psi_x(t)\right) ds \right)^2 dx \\
&\leq \left(\int_0^{+\infty} \frac{g^2(s)}{h(s)} ds \right) \int_0^1 \int_0^t h(t-s) \left(\psi_x(s) - \psi_x(t)\right)^2 ds dx \\
&= A_\alpha \left(h \circ \psi_x\right)(t).
\end{aligned} \tag{19}$$

□

Lemma 4 (Jensen's inequality)**.** *Given that G is a convex function on $[a,b]$, $f : \Omega \to [m,n]$ and h are integrable functions on Ω, $q(x) \geq 0$, and $\int_\Omega q(x)dx = \varrho > 0$, then*

$$G\left[\frac{1}{\varrho} \int_\Omega f(x)q(x)dx\right] \leq \frac{1}{\varrho} \int_\Omega G[f(x)]q(x)dx.$$

3. Strategic lemmas

For convenience, we will denote the norm $\|\cdot\|_{L^2(0,1)}$ and the inner product $\langle .,. \rangle_{L^2(0,1)}$ of the Lebesgue space $L^2(0,1)$ by $\|\cdot\|$ and $\langle .,. \rangle$ respectively. The constants $c > 0$ and $C > 0$ are generic constants which may change in value from one line to the other or within the same line. We define the energy functional of problem (7) as

$$
\begin{aligned}
E(t) =&\frac{\rho}{2}\|\varphi_t\|^2 + \frac{k}{2}\|\varphi_x\|^2 + \frac{J}{2}\|\psi_t\|^2 + \frac{a}{2}\|\psi\|^2 + \frac{1}{2}\left(\delta - \int_0^t g(s)ds\right)\|\psi_x\|^2 \\
&+ \frac{1}{2}(g\circ\psi_x)(t) + b\langle\varphi_x, \psi\rangle + \frac{\zeta}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\psi_t(s)\|^2 ds,
\end{aligned} \tag{20}
$$

where $\zeta > 0$ is a constant to be specified later, see [28] and $\lambda > 0$ satisfies

$$
0 < \lambda < \frac{2}{\tau_1}\log_e\left(\frac{\mu_1}{|\mu_2|}\sqrt{1-d}\right), \tag{21}
$$

and $(g\circ\psi_x)(t) = \int_0^t g(t-s)\|\psi_x(t) - \psi_x(s)\|^2 ds$.

Lemma 5. *Assume the conditions* $(A1) - (A5)$ *hold. Then, the energy functional* (20) *satisfies*

$$
\begin{aligned}
E'(t) \le& \frac{1}{2}(g'\circ\psi_x) - \frac{1}{2}g(t)\|\psi_x\|^2 - \left(\frac{\mu_1}{2} - \frac{\zeta}{2}\right)\|\psi_t\|^2 \\
&- \left[\frac{\zeta}{2}e^{-\lambda\tau_1}(1-d) - \frac{\mu_2^2}{2\mu_1}\right]\|\psi_t(t-\tau(t))\|^2 \\
&- \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\psi_t(s)\|^2 ds \ \le 0, \ \ \forall\, t \ge 0.
\end{aligned} \tag{22}
$$

Proof. Differentiation of (20) gives

$$
\begin{aligned}
E'(t) =&\rho\langle\varphi_t, \varphi_{tt}\rangle + k\langle\varphi_x, \varphi_{xt}\rangle + J\langle\psi_t, \psi_{tt}\rangle + a\langle\psi, \psi_t\rangle + b\frac{d}{dt}\langle\varphi_x, \psi\rangle \\
&+ \frac{1}{2}\frac{d}{dt}\left[\left(1 - \int_0^t g(s)ds\right)\|\psi_x(t)\|^2\right] + \frac{1}{2}\frac{d}{dt}(g\circ\psi_x)(t) \\
&+ \frac{\zeta}{2}\|\psi_t\|^2 - \frac{\zeta}{2}e^{-\lambda\tau(t)}(1-\tau'(t))\|\psi_t(t-\tau(t))\|^2 \\
&- \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\psi_t(s)\|^2 ds.
\end{aligned} \tag{23}
$$

Also, multiplying $(7)_1$ by φ_t, $(7)_2$ by ψ_t, integrating over $(0,1)$ and adding the two equations, we get

$$
\begin{aligned}
&\rho\langle\varphi_t, \varphi_{tt}\rangle + k\langle\varphi_x, \varphi_{xt}\rangle + J\langle\psi_t, \psi_{tt}\rangle + a\langle\psi, \psi_t\rangle + b\frac{d}{dt}\langle\varphi_x, \psi\rangle + \frac{1}{2}\frac{d}{dt}\left[\left(1 - \int_0^t g(s)ds\right)\|\psi_x(t)\|^2\right] + \frac{1}{2}\frac{d}{dt}(g\circ\psi_x)(t) \\
&= \frac{1}{2}(g'\circ\psi_x)(t) - \frac{1}{2}g(t)\|\psi_x(t)\|^2 - \mu_1\|\psi_t(t)\|^2 - \mu_2\langle\psi_t(t), \psi_t(t-\tau(t))\rangle.
\end{aligned} \tag{24}
$$

Young's inequality yields

$$
-\mu_2\langle\psi_t(t), \psi_t(t-\tau(t))\rangle \le \frac{\mu_1}{2}\|\psi_t\|^2 + \frac{\mu_2^2}{2\mu_1}\|\psi_t(t-\tau(t))\|^2. \tag{25}
$$

Substituting (24) into (23) and taking into account (25), assumptions (A3) and (A4), we get

$$
\begin{aligned}
E'(t) \le& \frac{1}{2}(g'\circ\psi_x)(t) - \frac{1}{2}g(t)\|\psi_x(t)\|^2 - \left(\frac{\mu_1}{2} - \frac{\zeta}{2}\right)\|\psi_t\|^2 \\
&- \left[\frac{\zeta}{2}e^{-\lambda\tau_1}(1-d) - \frac{\mu_2^2}{2\mu_1}\right]\|\psi_t(t-\tau(t))\|^2 - \frac{\lambda\zeta}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\psi_t(s)\|^2 ds.
\end{aligned} \tag{26}
$$

From condition (21), we can select $\zeta > 0$ so that

$$
\frac{\mu_2^2 e^{\lambda\tau_1}}{\mu_1(1-d)} < \zeta < \mu_1. \tag{27}
$$

Hence, (22) follows from (26) by virtue of $(A1)-(A2)$ and (27). This completes the proof. □

Lemma 6. *Let $t_0>0$. Then, the functional $F_1(t)=-J\int_0^1\psi_t\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)dsdx$ along the solution of (7) for any $\epsilon_1,\epsilon_2>0$ and $0<\alpha<1$ satisfies the estimate*

$$\begin{aligned} F_1'(t)\le &-\frac{Jg_0}{2}\|\psi_t\|^2+\epsilon_1\|\psi_x\|^2+\epsilon_2\|\varphi_x\|^2+\epsilon_3\|\psi\|^2+C\|\psi_t(t-\tau(t))\|^2\\ &+CA_\alpha\left(1+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}+\frac{1}{\epsilon_3}\right)(h\circ\psi_x)(t),\ \ \forall\, t\ge t_0. \end{aligned}\tag{28}$$

Proof. Differentiating F_1, using $(7)_2$ and integration by parts, we get

$$\begin{aligned} F_1'(t)=&-J\left(\int_0^t g(s)ds\right)\|\psi_t\|^2\underbrace{-J\int_0^1\psi_t\int_0^t g'(t-s)\left(\psi(t)-\psi(s)\right)dsdx}_{I_1}\\ &+\underbrace{\left(\delta-\int_0^t g(s)ds\right)\int_0^1\psi_x\int_0^t g(t-s)\left(\psi_x(t)-\psi_x(s)\right)dsdx}_{I_2}\\ &+\underbrace{\int_0^1\left(\int_0^t g(t-s)\left(\psi_x(t)-\psi_x(s)\right)ds\right)^2dx}_{I_3}+\underbrace{b\int_0^1\varphi_x\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)dsdx}_{I_4}\\ &+\underbrace{a\int_0^1\psi\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)dsdx}_{I_5}+\underbrace{\mu_1\int_0^1\psi_t\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)dsdx}_{I_6}\\ &+\underbrace{\mu_2\int_0^1\psi_t(t-\tau(t))\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)dsdx}_{I_7}. \end{aligned}\tag{29}$$

Using Cauchy-Schwarz, Young's and Poincaré's inequalities, Lemmas 1- 3 and similar computations as in (19), we estimate I_1-I_7 as follows:

$$\begin{cases} I_1=J\displaystyle\int_0^1\psi_t\int_0^t h(t-s)\left(\psi(t)-\psi(s)\right)dsdx\\ \quad-J\alpha\displaystyle\int_0^1\psi_t\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)dsdx\\ \quad\le\dfrac{\sigma_1}{2}\|\psi_t\|^2+\dfrac{C}{\sigma_1}\displaystyle\int_0^1\left(\int_0^t h(t-s)\left(\psi(t)-\psi(s)\right)ds\right)^2dx\\ \quad+\dfrac{C}{\sigma_1}\displaystyle\int_0^1\left(\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)ds\right)^2dx\\ \quad\le\dfrac{\sigma_1}{2}\|\psi_t\|^2+\dfrac{C}{\sigma_1}\left(\displaystyle\int_0^t h(s)ds\right)(h\circ\psi)(t)+\dfrac{CA_\alpha}{\sigma_1}(h\circ\psi)(t)\\ \quad\le\dfrac{\sigma_1}{2}\|\psi_t\|^2+\dfrac{C(A_\alpha+1)}{\sigma_1}(h\circ\psi_x)(t), & \text{for any } \sigma_1>0,\\ I_2\le\epsilon_1\|\psi_x\|^2+\dfrac{CA_\alpha}{\epsilon_1}(h\circ\psi_x)(t), & \text{for any } \epsilon_1>0,\\ I_3\le A_\alpha(h\circ\psi_x)(t),\\ I_4\le\epsilon_2\|\varphi_x\|^2+\dfrac{CA_\alpha}{\epsilon_2}(h\circ\psi_x)(t), & \text{for any } \epsilon_2>0,\\ I_5\le\epsilon_3\|\psi\|^2+\dfrac{CA_\alpha}{\epsilon_3}(h\circ\psi_x)(t), & \text{for any } \epsilon_3>0,\\ I_6\le\dfrac{\sigma_1}{2}\|\psi_t\|^2+\dfrac{CA_\alpha}{\sigma_1}(h\circ\psi_x)(t), & \text{for any } \sigma_1>0,\\ I_7\le\dfrac{\sigma_1}{2}\|\psi_t(t-\tau(t))\|^2+\dfrac{CA_\alpha}{\sigma_1}(h\circ\psi_x)(t), & \text{for any } \sigma_1>0. \end{cases}\tag{30}$$

Substituting the estimates in (30) into (29), we arrive at

$$F_1'(t) \leq -\left(J\int_0^t g(s)ds-\sigma_1\right)\|\psi_t\|^2+\epsilon_1\|\psi_x\|^2+\epsilon_2\|\varphi_x\|^2+\epsilon_3\|\psi\|^2 \\ +CA_\alpha\left(1+\frac{1}{\sigma_1}+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}+\frac{1}{\epsilon_3}\right)(h\circ\psi_x)(t)+\frac{\sigma_1}{2}\|\psi_t(t-\tau(t))\|^2. \quad (31)$$

From $(A1)$, we have that $g(0)>0$ and g is continuous. Therefore, for $t\geq t_0>0$, we obtain

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0. \quad (32)$$

Thus, we select $\sigma_1=\frac{Jg_0}{2}$ to get (28). This completes the proof. □

Lemma 7. *Let (φ,ψ) be the solution of Problem (7). Then, the functional $F_2(t)=-\rho\int_0^1\varphi_t\varphi dx$ satisfies the estimate*

$$F_2'(t)\leq-\rho\|\varphi_t\|^2+C\|\varphi_x\|^2+C\|\psi_x\|^2,\ \ \forall\, t\geq 0. \quad (33)$$

Proof. Differentiation of F_2, using $(7)_1$ and integration by parts, we obtain

$$F_2'(t)=-\rho\|\varphi_t\|^2+k\|\varphi_x\|^2+b\int_0^1\varphi_x\psi dx.$$

Applying Young's and Poincaré's inequalities, we obtain (33). This completes the proof. □

Lemma 8. *The functional $F_3(t)=J\int_0^1\psi_t\psi dx+\frac{b\rho}{k}\int_0^1\psi\int_0^x\varphi_t(y)dydx$ along the solution of problem (7) for any $\epsilon_4>0$ and $0<\alpha<1$ satisfies the estimate*

$$F_3'(t)=-\frac{l}{2}\|\psi_x\|^2-\left(a-\frac{b^2}{k}\right)\|\psi\|^2+\epsilon_4\|\varphi_t\|^2++C\left(1+\frac{1}{\epsilon_4}\right)\|\psi_t\|^2 \\ +CA_\alpha(h\circ\psi_x)(t)+C\|\psi_t(t-\tau(t))\|^2,\ \forall\, t\geq 0. \quad (34)$$

Proof. Differentiation of F_3, using (7) and integration by parts leads to

$$F_3'(t)=J\|\psi_t\|^2-\delta\|\psi_x\|^2-\left(a-\frac{b^2}{k}\right)\|\psi\|^2+\underbrace{\frac{b\rho}{k}\int_0^1\psi_t\int_0^x\varphi_t(y)dydx}_{I_8} \\ +\underbrace{\int_0^1\psi_x\int_0^t g(t-s)\psi_x(s)dsdx}_{I_9}-\underbrace{\mu_1\int_0^1\psi\psi_t dx}_{I_{10}}-\underbrace{\mu_2\int_0^1\psi\psi_t(t-\tau(t))dx}_{I_{11}}. \quad (35)$$

Using Cauchy-Schwarz, Young's and Poincaré's inequalities together with Lemmas 1− 3, we have

$$\begin{aligned} I_8 &\leq \epsilon_4\int_0^1\left(\int_0^x\varphi_t(y)dy\right)^2dx+\frac{(b\rho)^2}{4k^2\epsilon_4}\|\psi_t\|^2 \\ &\leq \epsilon_4\|\varphi_t\|^2+\frac{(b\rho)^2}{4k^2\epsilon_4}\|\psi_t\|^2,\ \text{ for any } \epsilon_4>0, \quad (36) \\ I_9 &= \int_0^1\psi_x\int_0^t g(t-s)\left(\psi_x(s)-\psi_x(t)\right)dsdx+\left(\int_0^t g(s)ds\right)\|\psi_x\|^2 \\ &\leq \left(\int_0^t g(s)ds+\frac{\sigma_2}{3}\right)\|\psi_x\|^2+\frac{C}{\sigma_2}\int_0^1\left(\int_0^t g(t-s)\left(\psi_x(s)-\psi_x(t)\right)ds\right)^2dx \\ &\leq \left(\int_0^t g(s)ds+\frac{\sigma_2}{3}\right)\|\psi_x\|^2+\frac{CA_\alpha}{\sigma_2}(h\circ\psi_x)(t),\ \text{ for any } \sigma_2>0, \quad (37) \\ I_{10} &\leq \frac{\sigma_2}{3}\|\psi_x\|^2+\frac{C\mu_1^2}{\sigma_2}\|\psi_t\|^2,\ \text{ for any } \sigma_2>0, \quad (38) \end{aligned}$$

and

$$I_{11} \leq \frac{\sigma_2}{3}\|\psi_x\|^2 + \frac{C\mu_2^2}{\sigma_2}\|\psi_t(t-\tau(t))\|^2, \text{ for any } \sigma_2 > 0. \tag{39}$$

Substituting (36)–(39) into (35), we arrive at

$$\begin{aligned} F_3'(t) = &-\left(\delta - \int_0^t g(s)ds - \sigma_2\right)\|\psi_x\|^2 - \left(a - \frac{b^2}{k}\right)\|\psi\|^2 + \epsilon_4\|\varphi_t\|^2 \\ &+ \left(J + \frac{(b\rho)^2}{4k^2\epsilon_4} + \frac{C\mu_1^2}{\sigma_2}\right)\|\psi_t\|^2 + \frac{CA_\alpha}{\sigma_2}\left(h\circ\psi_x\right)(t) + \frac{C\mu_2^2}{\sigma_2}\|\psi_t(t-\tau(t))\|^2. \end{aligned} \tag{40}$$

We choose $\sigma_2 = \frac{l}{2}$ to obtain (34). This completes the proof. □

Lemma 9. *Assume $\frac{k}{\rho} = \frac{\delta}{J}$. Then, the functional*

$$F_4(t) = \frac{|b|\delta\rho}{bk}\int_0^1 \varphi_t\psi_x dx + \frac{|b|J}{b}\int_0^1 \psi_t\varphi_x dx - \frac{|b|\rho}{bk}\int_0^1 \varphi_t\int_0^t g(t-s)\psi_x(s)dsdx,$$

along the solution of (7), for any $\epsilon_5 > 0$ and $0 < \alpha < 1$ satisfies the estimate

$$\begin{aligned} F_4'(t) \leq &-\frac{|b|}{2}\|\varphi_x\|^2 + \epsilon_5\|\varphi_t\|^2 + C\left(1 + \frac{1}{\epsilon_5}\right)\|\psi_x\|^2 + C\|\psi_t\|^2 + C\|\psi_t(t-\tau(t))\|^2 \\ &+ \frac{C(A_\alpha+1)}{\epsilon_5}\left(h\circ\psi_x\right)(t), \ \forall\, t \geq 0. \end{aligned} \tag{41}$$

Proof. Differentiating F_4, we get

$$\begin{aligned} F_4'(t) = &\frac{|b|\delta\rho}{bk}\int_0^1 \varphi_{tt}\psi_x dx + \frac{|b|\delta\rho}{bk}\int_0^1 \varphi_t\psi_{xt}dx + \frac{|b|J}{b}\int_0^1 \psi_{tt}\varphi_x dx \\ &+ \frac{|b|J}{b}\int_0^1 \psi_t\varphi_{xt}dx - \frac{|b|\rho}{bk}\int_0^1 \varphi_{tt}\int_0^t g(t-s)\psi_x(s)dsdx \\ &- \frac{|b|\rho}{bk}\int_0^1 \varphi_t\int_0^t g'(t-s)\psi_x(s)dsdx - \frac{|b|\rho}{bk}g(0)\int_0^1 \varphi_t\psi_x dx. \end{aligned}$$

Using these equations in (7) and integration by parts, we arrive at

$$\begin{aligned} F_4'(t) = &\frac{|b|\delta\rho}{k}\|\psi_x\|^2 + \frac{|b|}{b}\left(\frac{\delta\rho}{k} - J\right)\int_0^1 \varphi_x\psi_{xt}dx - |b|\|\varphi_x\|^2 \\ &- \frac{|b|\mu_1}{b}\int_0^1 \psi_t\varphi_x dx - \frac{|b|\mu_2}{b}\int_0^1 \psi_t(t-\tau(t))\varphi_x dx \\ &- \frac{|b|a}{b}\int_0^1 \psi\varphi_x dx - \frac{|b|}{k}\int_0^1 \psi_x\int_0^t g(t-s)\psi_x(s)dsdx \\ &- \frac{|b|\rho}{bk}\int_0^1 \varphi_t\int_0^t g'(t-s)\psi_x(s)dsdx - \frac{|b|\rho}{bk}g(0)\int_0^1 \varphi_t\psi_x dx. \end{aligned}$$

Using the fact that $\frac{k}{\rho} = \frac{\delta}{J}$, we get

$$\begin{aligned} F_4'(t) = &-|b|\|\varphi_x\|^2 + \frac{|b|\delta\rho}{k}\|\psi_x\|^2 - \underbrace{\frac{|b|a}{b}\int_0^1 \psi\varphi_x dx}_{G_1} - \underbrace{\frac{|b|\mu_1}{b}\int_0^1 \psi_t\varphi_x dx}_{G_2} \\ &- \underbrace{\frac{|b|\mu_2}{b}\int_0^1 \psi_t(t-\tau(t))\varphi_x dx}_{G_3} - \underbrace{\frac{|b|}{k}\int_0^1 \psi_x\int_0^t g(t-s)\psi_x(s)dsdx}_{G_4} \\ &- \underbrace{\frac{|b|\rho}{bk}\int_0^1 \varphi_t\int_0^t g'(t-s)\psi_x(s)dsdx}_{G_5} - \underbrace{\frac{|b|\rho}{bk}g(0)\int_0^1 \varphi_t\psi_x dx}_{G_6}. \end{aligned} \tag{42}$$

Applying Cauchy-Schwarz, Young's and Poincaré's inequalities, taking into account Lemmas 1– 3, $h = \alpha g - g'$, we have for any $\sigma_3, \epsilon_5 > 0$

$$G_1 \leq \frac{\sigma_3}{3}\|\varphi_x\|^2 + \frac{C}{\sigma_3}\|\psi_x\|^2, \tag{43}$$

$$G_2 \leq \frac{\sigma_3}{3}\|\varphi_x\|^2 + \frac{C}{\sigma_3}\|\psi_t\|^2, \tag{44}$$

$$G_3 \leq \frac{\sigma_3}{3}\|\varphi_x\|^2 + \frac{C}{\sigma_3}\|\psi_t(t-\tau(t))\|^2, \tag{45}$$

$$\begin{aligned} G_4 \leq& \frac{|b|}{2k}\|\psi_x\|^2 + \frac{|b|}{2k}\int_0^1 \left(\int_0^t g(t-s)\psi_x(s) - \psi_x(t)ds + \int_0^t g(s)ds\psi_x(t)\right)^2 dx \\ \leq& \frac{|b|}{2k}\|\psi_x\|^2 + \frac{|b|}{k}\int_0^1 \left(\int_0^t g(t-s)\psi_x(s) - \psi_x(t)ds\right)^2 dx + \frac{|b|(\delta-l)^2}{2k}\|\psi_x\|^2 \\ \leq& \left(\frac{|b|}{2k} + \frac{|b|(\delta-l)^2}{k}\right)\|\psi_x\|^2 + \frac{|b|}{k}A_\alpha\left(h\circ\psi_x\right)(t), \end{aligned} \tag{46}$$

$$\begin{aligned} G_5 =& \frac{|b|\rho}{bk}\int_0^1 \varphi_t \int_0^t h(t-s)(\psi_x(s) - \psi_x(t))dsdx + \frac{|b|\rho}{bk}\left(\int_0^t h(s)ds\right)\int_0^1 \varphi_t\psi_x dx \\ &- \frac{|b|\rho\alpha}{bk}\int_0^1 \varphi_t \int_0^t g(t-s)\psi_x(s)dsdx \\ \leq& \frac{3\epsilon_5}{4}\|\varphi_t\|^2 + \frac{C}{\epsilon_5}\int_0^1\left(\int_0^t h(t-s)\psi_x(s) - \psi_x(t)ds\right)^2 dx + \frac{C}{\epsilon_5}\|\psi_x\|^2 \\ &+ \frac{C}{\epsilon_5}\int_0^1\left(\int_0^t g(t-s)\psi_x(s) - \psi_x(t)ds + \int_0^t g(s)ds\psi_x(t)\right)^2 dx \\ \leq& \frac{3\epsilon_5}{4}\|\varphi_t\|^2 + \frac{C}{\epsilon_5}\|\psi_x\|^2 + \frac{C(A_\alpha+1)}{\epsilon_5}\left(h\circ\psi_x\right)(t), \end{aligned} \tag{47}$$

and

$$G_6 \leq \frac{\epsilon_5}{4}\|\varphi_t\|^2 + \frac{C}{\epsilon_5}\|\psi_x\|^2. \tag{48}$$

Substitution of (43) – (48) into (42) yields

$$\begin{aligned} F_4'(t) =& -\left(|b| - \sigma_3\right)\|\varphi_x\|^2 + \left(\frac{|b|\delta\rho}{k} + \frac{C}{\sigma_3} + \frac{|b|}{2k} + \frac{|b|(\delta-l)^2}{k} + \frac{C}{\epsilon_5}\right)\|\psi_x\|^2 \\ &+ \epsilon_5\|\varphi_t\|^2 + \frac{C}{\sigma_3}\|\psi_t\|^2 + \frac{C}{\sigma_3}\|\psi_t(t-\tau(t))\|^2 + \left(CA_\alpha + \frac{C(A_\alpha+1)}{\epsilon_5}\right)\left(h\circ\psi_x\right)(t). \end{aligned} \tag{49}$$

Finally, we choose $\sigma_3 = \dfrac{|b|}{2}$ to obtain (41). This completes the proof. □

Lemma 10. *The functional* $F_5(t) = \int_0^t f(t-s)\|\psi_x(s)\|^2 ds$, *where* $f(t) = \displaystyle\int_t^{+\infty} g(s)ds$, *along the solution of* (7) *satisfies the estimate*

$$F_5'(t) \leq 3(\delta - l)\|\psi_x\|_2^2 - \frac{1}{2}(g\circ\psi_x)(t), \ \forall\, t \geq 0. \tag{50}$$

Proof. Differentiation of F_5 and using the fact that $f'(t) = -g(t)$ lead to

$$\begin{aligned} F_5'(t) =& \int_0^t f'(t-s)\|\psi_x(s)\|^2 ds + f(0)\|\psi_x\|^2 \\ =& -\left(g\circ\psi_x\right)(t) + f(t)\|\psi_x\|^2 - 2\int_0^1 \psi_x \int_0^t g(t-s)\left(\psi_x(s) - \psi_x(t)\right)dsdx. \end{aligned}$$

Cauchy-Schwarz inequality and condition $(A1)$ give

$$-2\int_0^1 \psi_x \int_0^t g(t-s)\left(\psi_x(s) - \psi_x(t)\right)dsdx \leq 2(\delta - l)\|\psi_x\|^2 + \frac{\int_0^t g(s)ds}{2(\delta-l)}(g\circ\psi_x)(t) \leq 2(\delta-l)\|\psi_x\|^2 + \frac{1}{2}(g\circ\psi_x)(t).$$

Therefore,

$$F_5'(t) \leq 2(\delta - l)\|\psi_x\|^2 - \frac{1}{2}(g \circ \psi_x)(t) + f(t)\|\psi_x\|^2.$$

Since $f'(t) = -g(t) \leq 0$, it follows that $f(t) \leq f(0) = \delta - l$. Thus, we have

$$F_5'(t) \leq 3(\delta - l)\|\psi_x\|^2 - \frac{1}{2}(g \circ \psi_x)(t), \ \forall\, t \geq 0.$$

□

For the next lemma, we consider the Lyapunov functional K defined by

$$K(t) = NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + N_4F_4(t), \tag{51}$$

where $N,\ N_j,\ j = 1,2,3,4$ are positive constants to be specified later.

Lemma 11. *Assume $\frac{k}{\rho} = \frac{\delta}{J}$. Then, for suitable choice of $N,\ N_j,\ j = 1,2,3,4$, the Lyapunov functional K along the solution of (7) satisfies the estimate*

$$K'(t) \leq -\beta\left(\|\varphi_t\|^2 + \|\varphi_x\|^2 + \|\psi_t\|^2 + \|\psi_x\|^2 + \|\psi\|^2\right) + \frac{1}{4}\left(g \circ \psi_x\right)(t), \forall\, t \geq t_0 \tag{52}$$

for some $\beta > 0$ and $K \sim E$, that is

$$\alpha_1 E(t) \leq K(t) \leq \alpha_2 E(t),\ \forall\, t \geq 0 \tag{53}$$

holds for some $\alpha_1,\ \alpha_2 > 0$.

Proof. Using (51) and recalling that $h(t) = \alpha g(t) - g'(t)$, then Lemmas 6–10 yields, for all $t \geq t_0$,

$$\begin{aligned}
K'(t) \leq & -\left[N_2\rho - N_3\epsilon_4 - N_4\epsilon_5\right]\|\varphi_t\|^2 - \left[\frac{N_4|b|}{2} - N_2C - N_1\epsilon_2\right]\|\varphi_x\|^2 \\
& - \left[N\gamma_2 + \frac{N_1Jg_0}{2} - N_3\left(1 + \frac{1}{\epsilon_4}\right) - N_4C\right]\|\psi_t\|^2 \\
& - \left[\frac{N_3l}{2} - N_2C - N_1\epsilon_1 - N_4C\left(1 + \frac{1}{\epsilon_5}\right)\right]\|\psi_x\|^2 - \left[N_3\gamma_1 - N_1\epsilon_3\right]\|\psi\|^2 + \frac{N\alpha}{2}\left(g \circ \psi_x\right)(t) \\
& - \left[N\gamma_3 - N_1C - N_3C - N_4C\right]\|\psi_t(t - \tau(t))\|^2 - \frac{N\lambda\zeta}{2}\int_{t-\tau(t)}^{t} e^{-\lambda(t-s)}\|\psi_t(s)\|^2 ds \\
& - \left[\frac{N}{2} - CA_\alpha\left(N_1\left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{1}{\epsilon_3}\right) + N_3 + \frac{N_4}{\epsilon_5}\right)\right]\left(h \circ \psi_x\right)(t),
\end{aligned} \tag{54}$$

where $\gamma_1 = \left(a - \frac{b^2}{k}\right) > 0,\ \gamma_2 = \left(\frac{\mu_1}{2} - \frac{\zeta}{2}\right) > 0,\ \gamma_3 = \left(\frac{\zeta}{2}e^{-\lambda\tau_1}(1 - d) - \frac{\mu_2^2}{2\mu_1}\right) > 0$ by virtue of (3) and (27). Next, we choose

$$N_2 = 1,\ \epsilon_1 = \frac{N_3l}{4N_1},\ \epsilon_2 = \frac{N_4|b|}{4N_1},\ \epsilon_3 = \frac{N_3\gamma_1}{2N_2},\ \epsilon_4 = \frac{\rho}{4N_3},\ \epsilon_5 = \frac{\rho}{4N_4} \tag{55}$$

and (54) becomes

$$\begin{aligned}
K'(t) \leq & -\frac{\rho}{2}\|\varphi_t\|^2 - \left[\frac{N_4|b|}{4} - C\right]\|\varphi_x\|^2 - \left[N\gamma_2 + \frac{N_1Jg_0}{2} - N_3\left(1 + \frac{4N_3}{\rho}\right) - N_4C\right]\|\psi_t\|^2 \\
& - \left[\frac{N_3l}{4} - N_4C\left(1 + \frac{4N_4}{\rho}\right) - C\right]\|\psi_x\|^2 - \frac{N_3\gamma_1}{2}\|\psi\|^2 \\
& - \left[N\gamma_3 - (N_1 + N_3 + N_4)C\right]\|\psi_t(t - \tau(t))\|^2 \\
& + \frac{N\alpha}{2}\left(g \circ \psi_x\right)(t) - \frac{N\lambda\zeta}{2}\int_{t-\tau(t)}^{t} e^{-\lambda(t-s)}\|\psi_t(s)\|^2 ds \\
& - \left[\frac{N}{2} - CA_\alpha\left(N_1\left(1 + \frac{4N_1}{N_3l} + \frac{4N_1}{N_4|b|} + \frac{2N_1}{N_3\gamma_1}\right) + N_3 + \frac{N_4^2}{\rho}\right)\right]\left(h \circ \psi_x\right)(t).
\end{aligned}$$

Now, we choose the remaining constants: First, we select N_4 so that

$$\frac{N_4|b|}{4} - C > 0. \tag{56}$$

Then, we choose N_3 large enough such that

$$\frac{N_3 l}{4} - N_4 C\left(1 + \frac{4N_4}{\rho}\right) - C > 0. \tag{57}$$

Hence N_3 and N_4 are fixed, we choose N_1 large so that

$$\frac{N_1 J g_0}{2} - N_3\left(1 + \frac{4N_3}{\rho}\right) - N_4 C > 0. \tag{58}$$

We have that $\frac{\alpha g^2(s)}{h(s)} = \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s)$. Thus, using the dominated convergence theorem, we get

$$\lim_{\alpha\to 0} \alpha A_\alpha = \lim_{\alpha\to 0} \int_0^{+\infty} \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} ds = 0. \tag{59}$$

Thus, there exist $0 < \alpha_0 < 1$ such that for all $0 < \alpha \le \alpha_0$, we have

$$\alpha A_\alpha < \frac{1}{4C\left(N_1\left(1 + \frac{4N_1}{N_3 l} + \frac{4N_1}{N_4|b|} + \frac{2N_1}{N_3\gamma_1}\right) + N_3 + \frac{N_4^2}{\rho}\right)}. \tag{60}$$

Finally, we choose N so large and take $\alpha = \frac{1}{2N}$ Such that

$$\begin{aligned} &N\gamma_3 - (N_1 + N_3 + N_4)C > 0, \\ &\frac{N}{2} - CA_\alpha\left(N_1\left(1 + \frac{4N_1}{N_3 l} + \frac{4N_1}{N_4|b|} + \frac{2N_1}{N_3\gamma_1}\right) + N_3 + \frac{N_4^2}{\rho}\right) > 0. \end{aligned} \tag{61}$$

The analysis from (55) $-$ (61) yields (52). Applying Young's, Cauchy-Schwarz, and Poincaré's inequalities, we obtain (53) easily. This completes the proof. □

4. Main stability result

The main stability result of the work is the following:

Theorem 2. *Assume* $\frac{k}{\rho} = \frac{\delta}{J}$ *and* $(A1) - (A5)$ *hold. Then, there exist* $\lambda_1 > 0, \lambda_2 > 0$ *such that the solution energy* (20) *satisfies*

$$E(t) \le \lambda_2 M_1^{-1}\left(\lambda_1 \int_{t_0}^t \xi(s)ds\right), \ \forall\, t \ge t_0, \tag{62}$$

where $M_1(t) = \int_t^r \frac{1}{sM'(s)} ds$ *and* M_1 *is a strictly decreasing and strictly convex function on* $(0, r]$ *with* $\lim_{t\to 0} M_1(t) = +\infty$.

Proof. From (13) and (22), we have $\forall\, t \ge t_0$

$$\int_0^{t_0} g(s)\|\psi_x(t) - \psi_x(t-s)\|^2 ds \le -\frac{g(0)}{\beta_1}\int_0^{t_0} g'(s)\|\psi_x(t) - \psi_x(t-s)\|^2 ds \le -CE'(t). \tag{63}$$

Thus, (52) and (63) implies

$$K'(t) \le -\eta E(t) + \frac{1}{2}(g\circ\psi_x)(t) \le -\eta E(t) - CE'(t) + \frac{1}{2}\int_{t_0}^t g(s)\|\psi_x(t) - \psi_x(t-s)\|^2 ds,$$

for some $\eta > 0$. Let $K_1 = K + CE \sim E$, by virtue of (53), it follows that

$$K_1'(t) \le -\eta E(t) + \frac{1}{2}\int_{t_0}^t g(s)\|\psi_x(t) - \psi_x(t-s)\|^2 ds, \ \forall\, t \ge t_0. \tag{64}$$

To complete the proof, we divide it into two parts:
Case 1: when $M(t)$ is linear. Multiplying (64) by $\xi(t)$, keeping in mind (22) and $(A2)$, we get

$$\begin{aligned}\xi(t)K_1'(t) &\leq -\eta\xi(t)E(t) + \frac{1}{2}\xi(t)\int_{t_0}^{t} g(s)\|\psi_x(t)-\psi_x(t-s)\|^2 ds \\ &\leq -\eta\xi(t)E(t) + \frac{1}{2}\int_{t_0}^{t}\xi(s)g(s)\|\psi_x(t)-\psi_x(t-s)\|^2 ds \\ &\leq -\eta\xi(t)E(t) - \frac{1}{2}\int_{t_0}^{t} g'(s)\|\psi_x(t)-\psi_x(t-s)\|^2 ds \\ &\leq -\eta\xi(t)E(t) - CE'(t),\ \forall\, t \geq t_0. \end{aligned} \tag{65}$$

From (A2), ξ is non-increasing, thus, we get

$$(\xi K_1 + CE)'(t) \leq -\eta\xi(t)E(t), \quad \forall\, t \geq t_0, \tag{66}$$

and

$$\xi K_1 + CE \sim E. \tag{67}$$

Since $K \sim E$, thus, setting $K_2(t) = \xi(t)K_1(t) + CE(t)$, there exists $\eta_1 > 0$ so such that

$$K_2'(t) \leq -\eta\xi(t)E(t) \leq -\eta_1\xi(t)K_2(t), \ \forall\, t \geq t_0. \tag{68}$$

Integration of (68) over (t_0, t) and recalling (67), we obtain

$$E(t) \leq \lambda_1 e^{-\lambda_2\int_{t_0}^{t}\xi(s)ds} = \lambda_1 M_1^{-1}\left(\lambda_2\int_{t_0}^{t}\xi(s)ds\right).$$

Case 2: when $M(t)$ is nonlinear. In this case, we consider $\mathcal{K}(t) = K(t) + F_5(t)$. Then, Lemmas 10 and (52) yield for some $d > 0$

$$\mathcal{K}'(t) \leq -dE(t), \ \forall\, t \geq t_0. \tag{69}$$

It follows that $d\int_{t_0}^{t} E(s)ds \leq \mathcal{K}(t_0) - \mathcal{K}(t) \leq \mathcal{K}(t_0)$, from which we get

$$\int_0^{+\infty} E(s)ds < \infty. \tag{70}$$

From (70), we can define $p(t)$ by $p(t) := \varpi\int_{t_0}^{t}\|\psi_x(t)-\psi_x(t-s)\|^2 ds$, where $0 < \varpi < 1$ so that

$$p(t) < 1, \forall\, t \geq t_0. \tag{71}$$

Furthermore, we assume $p(t) > 0$ for all $t \geq t_0$; otherwise, it follows from (64) that the solution energy is exponentially stable. Also, we define the functional $q(t)$ by $q(t) = -\int_{t_0}^{t} g'(s)\|\psi_x(t)-\psi_x(t-s)\|^2 ds$ and it's easy to see that $q(t) \leq -CE'(t),\ \forall\, t \geq t_0$. From (A_2), M is strictly convex on $(0, r]$ (where $r = g(t_0)$) and $M(0) = 0$, this implies

$$M(\theta s) \leq \theta M(s), \ \ 0 \leq \theta \leq 1 \ \text{ and } s \in (0, r]. \tag{72}$$

Using (71),(72), assumption $(A2)$ and Jensen's inequality, we have

$$\begin{aligned} q(t) &= \frac{1}{\varpi p(t)}\int_{t_0}^{t} p(t)(-g'(s))\varpi\|\psi_x(t)-\psi_x(t-s)\|^2 ds \\ &\geq \frac{1}{\varpi p(t)}\int_{t_0}^{t} p(t)\xi(s)M(g(s))\varpi\|\psi_x(t)-\psi_x(t-s)\|^2 ds \\ &\geq \frac{\xi(t)}{\varpi p(t)}\int_{t_0}^{t} M(p(t)g(s))\varpi\|\psi_x(t)-\psi_x(t-s)\|^2 ds \\ &\geq \frac{\xi(t)}{\varpi}M\left(\varpi\int_{t_0}^{t} g(s)\|\psi_x(t)-\psi_x(t-s)\|s^2 ds\right) \end{aligned}$$

$$= \frac{\xi(t)}{\varpi}\bar{M}\left(\varpi\int_{t_0}^{t} g(s)\|\psi_x(t)-\psi_x(t-s)\|^2 ds\right), \tag{73}$$

where $\bar{M}$ is the convex extension of M on $(0,+\infty)$, see (14). From (73), we have

$$\int_{t_0}^{t} g(s)\|\psi_x(t)-\psi_x(t-s)\|^2 ds \leq \frac{1}{\varpi}\bar{M}^{-1}\left(\frac{\varpi q(t)}{\xi(t)}\right).$$

Thus, (64) gives

$$K_1'(t) \leq -\eta E(t) + C\bar{M}^{-1}\left(\frac{\varpi q(t)}{\xi(t)}\right), \ \forall\, t \geq t_0. \tag{74}$$

Let $r_0 < r$ to be specified later and define the functional K_3 by $K_3(t) := \bar{M}'\left(r_0\frac{E(t)}{E(0)}\right)K_1(t) + E(t) \sim E(t)$. Since $K_1 \sim E$. Using (74) and recalling that $E'(t) \leq 0,\ \bar{M}'(t) > 0,\ \bar{M}''(t) > 0$, we have for all $t \geq t_0$

$$\begin{aligned} K_3'(t) &= r_0\frac{E'(t)}{E(0)}\bar{M}''\left(r_0\frac{E(t)}{E(0)}\right)K_1(t) + \bar{M}'\left(r_0\frac{E(t)}{E(0)}\right)K_1'(t) + E'(t) \\ &\leq -\eta E(t)\bar{M}'\left(r_0\frac{E(t)}{E(0)}\right) + C\bar{M}'\left(r_0\frac{E(t)}{E(0)}\right)\bar{M}^{-1}\left(\varpi\frac{q(t)}{\xi(t)}\right) + E'(t). \end{aligned} \tag{75}$$

Now, we consider conjugate of $\bar{M}$ denoted by $\bar{M}^*$ define in the sense of Young, see Arnold [29] page 61-64, define by

$$\bar{M}^*(s) = s(\bar{M}')^{-1}(s) - \bar{M}\left[(\bar{M}')(s)\right] \tag{76}$$

and $\bar{M}^*$ satisfies the Young's inequality

$$XY \leq \bar{M}^*(X) + \bar{M}(Y). \tag{77}$$

Setting $X = \bar{M}'\left(r_0\frac{E(t)}{E(0)}\right)$ and $Y = \bar{M}^{-1}\left(\varpi\frac{q(t)}{\xi(t)}\right)$, then (22) and (75)−(77) yield for all $t \geq t_0$

$$\begin{aligned} K_3'(t) &\leq -\eta E(t)\bar{M}'\left(r_0\frac{E(t)}{E(0)}\right) + C\bar{M}^*\left(\bar{M}'\left(r_0\frac{E(t)}{E(0)}\right)\right) + C\varpi\frac{q(t)}{\xi(t)} + E'(t) \\ &\leq -\eta E(t)\bar{M}'\left(r_0\frac{E(t)}{E(0)}\right) + Cr_0\frac{E(t)}{E(0)}\bar{M}'\left(r_0\frac{E(t)}{E(0)}\right) + C\varpi\frac{q(t)}{\xi(t)} + E'(t). \end{aligned} \tag{78}$$

Now, multiplying (78) by $\xi(t)$ keeping in mind that $r_0\frac{E(t)}{E(0)} < r$ and $\bar{M}'\left(r_0\frac{E(t)}{E(0)}\right) = M'\left(r_0\frac{E(t)}{E(0)}\right)$, we arrive at

$$\begin{aligned} \xi(t)K_3'(t) &\leq -\eta\xi(t)E(t)M'\left(r_0\frac{E(t)}{E(0)}\right) + Cr_0\frac{E(t)}{E(0)}\xi(t)M'\left(r_0\frac{E(t)}{E(0)}\right) + C\varpi q(t) + \xi(t)E'(t) \\ &\leq -\eta\xi(t)E(t)M'\left(r_0\frac{E(t)}{E(0)}\right) + Cr_0\frac{E(t)}{E(0)}\xi(t)M'\left(r_0\frac{E(t)}{E(0)}\right) - CE'(t). \end{aligned} \tag{79}$$

We set $K_4(t) = \xi(t)K_3(t) + CE(t) \sim E(t)$ since $K_3 \sim E$. Thus there exist n_0, n_1 positive such that

$$n_0 K_4(t) \leq E(t) \leq n_1 K_4(t). \tag{80}$$

Therefore, estimate (79) yields

$$K_4'(t) \leq -(\eta E(0) - Cr_0)\xi(t)\frac{E(t)}{E(0)}\xi(t)M'\left(r_0\frac{E(t)}{E(0)}\right), \ \forall\, t \geq t_0.$$

We choose $r_0 < r$ small enough so that $\eta E(0) - Cr_0 > 0$ to arrive at

$$K_4'(t) \leq -\eta_2\xi(t)\frac{E(t)}{E(0)}\xi(t)M'\left(r_0\frac{E(t)}{E(0)}\right) = -\eta_2\xi(t)M_2\left(\frac{E(t)}{E(0)}\right), \ \ \forall\, t \geq t_0, \tag{81}$$

for some constant $\eta_2 > 0$ and $M_2(s) = sM'(r_0 s)$. We note that $M_2'(s) = M'(r_0 s) + r_0 s M''(r_0 s)$, hence, the strict convexity of M on $(0, r]$, yields $M_2(s) > 0$, $M_2'(s) > 0$ on $(0, r]$. Let $R(t) = n_0 \frac{K_4(t)}{E(0)}$. Using (80) and (81), we obtain

$$R(t) \sim E(t), \tag{82}$$

and

$$R'(t) = n_0 \frac{K_4'(t)}{E(0)} \leq -\eta_3 \xi(t) M_2(R(t)), \ \forall\, t \geq t_0, \tag{83}$$

for some $\eta_3 > 0$. Integration of (83) over (t_0, t), gives

$$\eta_3 \int_{t_0}^{t} \xi(s) ds \leq - \int_{t_0}^{t} \frac{R'(s)}{M_2(R(s))} ds = \frac{1}{r_0} \int_{r_0 R(t)}^{r_0 R(t_0)} \frac{1}{sM'(s)} ds, \tag{84}$$

from which we get

$$R(t) \leq \frac{1}{r_0} M_1^{-1} \left(\eta_3 \int_{t_0}^{t} \xi(s) ds \right), \quad \text{where } M_1(t) = \int_t^r \frac{1}{sM'(s)} ds. \tag{85}$$

Using properties of M, we easily see that M_1 is strictly decreasing function on $(0, r]$ and $\lim_{t \longrightarrow 0} M_1(t) = +\infty$. Therefore, (62) follows from (82) and (85). This completes the proof. □

Remark 1. The stability result in Theorem 2 is general and optimal in the sense that it agrees with the decay rate of g, see [30], Remark 2.3.

Corollary 1. *Suppose $\frac{k}{\rho} = \frac{\delta}{J}$, and $(A1) - (A5)$ hold. Assume the function M in (A_2) be defined by $H(s) = s^p$, $p \geq 1$, then there exist $\lambda_1, \lambda_2, C > 0$ such that (20) satisfies*

$$E(t) \leq \begin{cases} \lambda_2 \exp\left(-\lambda_1 \int_0^t \xi(s) ds \right), & \text{for } p = 1, \\ \dfrac{C}{\left(1 + \int_{t_0}^{t} \xi(s) ds \right)^{\frac{1}{p-1}}}, & \text{for } p > 1. \end{cases} \tag{86}$$

5. Examples

(1). Let $g(t) = \nu_1 e^{-\nu_2 t}$, $t \geq 0$, $\nu_1, \nu_2 > 0$ and ν_2 is chosen so that (A_1) holds. Then, $g'(t) = -\nu_1 \nu_2 e^{-\nu_2 t} = -\nu_2 M(g(t))$ with $M(t) = t$. Thus, from (62), the solution energy (20) satisfies $E(t) \leq Ce^{-\lambda t}$, $\forall\, t \geq 0$, where $\lambda = \nu_2 \lambda_1$.

(2). Let $g(t) = ue^{-(1+t)^v}$, $t \geq 0$, $u > 0$, $0 < v < 1$ are constants and u is chosen such that (A_1) holds. Then, $g'(t) = -uv(1+t)^{v-1} e^{-(1+t)^v} = -\xi(t) M(g(t))$, where $\xi(t) = v(1+t)^{v-1}$ and $M(t) = t$. Thus, we get from (62) that $E(t) \leq \lambda_2 e^{-\lambda_1 (1+t)^v}$, $\forall\, t \geq 0$.

(3). Let $g(t) = \frac{u}{(1+t)^v}$, $t \geq 0$, $u > 0$, $v > 1$ are constants and u is chosen such that (A_1) holds. We have $g'(t) = \frac{-uv}{(1+t)^{v+1}} = -\xi \left(\frac{u}{(1+t)^v} \right)^{\frac{v+1}{v}} = -\xi g^p(t) = -\xi M(g(t))$, where $M(t) = t^p$, $p = \frac{v+1}{v}$ satisfying $1 < p < 2$ and $\xi = \frac{v}{u^{\frac{1}{v}}} > 0$. Hence, we deduce from (86) that $E(t) \leq \frac{C}{(1+t)^v}$, $\forall\, t \geq 0$.

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